Graded roots and singularities

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Contents

1 Introduction

The present article has a two-fold goal. Firstly, it aims to advertise the graded roots introduced by the author in his study of the topology of normal surface singularities. In the body of the paper we emphasize two aspects of them: their potential role in the classification of normal surface singularities, and also their connections with the Seiberg–Witten (and Heegaard–Floer) theory of rational homology sphere 3–manifolds.

In order to make the presentation more complete, we organized some of the sections in the spirit of a review article, listing some of the important constructions and results relevant to the theory. In these sections there are less proofs, but we always provide the original sources.

As a second goal of the article, we provide also a series of new results. For example, sections 4 and 5 consist of unpublished results (although they circulated in preprint form, cf. [35, 36]). These sections contain all the proofs with all the necessary details.

The main motivation of the author in the connection of singularity theory with the Seiberg-Witten invariant of 3–manifolds was born when the article [37] was written: this article formulates a conjecture connecting topological and analytical invariants (for its generalization, see section 4). For the 'history' and 'realizations' of this conjecture the reader is invited to read [33]. In the first verifications of the conjecture for different particular families, we used the realization of the Seiberg-Witten invariants based on the Turaev's torsion. Later, the article [51] provided a different model how one can understand these invariants via Heegaard–Floer homology. It was a big surprise for the author that some parts of the computational technique described in [51] for the Heegaard–Floer homology resonated incredibly with the technique of computational sequences initiated by Laufer and S. S.-T. Yau (and used by author too) in the computation of different singularity invariants (like the geometric genus and Hilbert–Samuel function). This alloy lead to the definition of graded roots, and to the algorithms of its computations (for almost rational plumbing graphs, a family which is also a novelty of the theory).

We definitely believe that the theory of graded roots will have many deep applications. In fact, we believe that it is the guiding structure in many phenomena. This is exemplified in 3.4, but also in the last section by its appearance in a different problem, which a priori sits rather far from the above circle of ideas, the classification of rational unicuspidal projective plane curves.

For the organization of the article, see contents above.

2 Normal surface singularities.

2.1 The link.

2.1.1. Definition. The link. Let $(X, 0)$ be a complex analytic normal surface singularity embedded in $(\mathbb{C}^N, 0)$, and let B_ϵ be the ϵ-ball in $\mathbb{C}^N$ centered at the origin. Then, for ϵ sufficiently small, the intersection $M := X \cap \partial B_\epsilon$ is a connected compact oriented 3-manifold, whose oriented C^∞ type does not depend on the choice of the embedding and ϵ. It is called the *link* of $(X, 0)$ [28]. Moreover, $X \cap B_\epsilon$ is homeomorphic to the cone over M. In particular, M characterizes completely the local topological type of $(X, 0)$. Therefore, if an invariant of $(X, 0)$ can be deduced from M, we say that it is a *topological invariant*.

2.1.2. The link as a plumbed manifold. Not any oriented 3-manifold can be realized as the link of a singularity. In order to see this, consider the following resolution procedure. Fix a sufficiently small Stein representative X of $(X, 0)$ (e.g. $X \cap B_\epsilon$ as above) and let $\pi : \tilde{X} \to X$ be a resolution of the singular point $0 \in X$. In particular, $\tilde{X}$ is smooth, and π is a biholomorphic isomorphism above $X \setminus \{0\}$. We will assume that the exceptional divisor $E := \pi^{-1}(0)$ is a normal crossing divisor with irreducible components $\{E_j\}_{j\in\mathcal{J}}$. Such a resolution is called *good*. For a good resolution π, let $\Gamma(\pi)$ be the dual resolution graph associated with π decorated with the self intersection numbers $\{(E_j, E_j)\}_j$ and genera $\{g_j\}_{j\in\mathcal{J}}$ (see [19]). Sometimes we write e_j for (E_j, E_j). Notice that $H_2(\tilde{X}, \mathbb{Z})$ is freely generated by the fundamental classes $\{[E_j]\}_j$. Let I be the intersection matrix $\{(E_j, E_i)\}_{j,i}$. Since π identifies $\partial\tilde{X}$ with M, the graph $\Gamma(\pi)$ can be regarded a plumbing graph, and M can be considered as an S^1-plumbed manifold whose plumbing graph is $\Gamma(\pi)$.

The crucial point is that $\Gamma(\pi)$ is connected and I is negative definite [29]. The converse is also true, it was proved by Grauert [12]: a connected plumbing graph can be realized as a resolution graph of a (complex analytic) normal surface singularity if and only if the associated intersection form I is negative definite. This gives a complete classification of the possible topological types of (analytic) normal surface singularities.

2.1.3. Assumption. We note that M is a *rational homology sphere* ($\mathbb{Q}HS$), i.e. $H_1(M, \mathbb{Q}) = 0$, if and only if $\Gamma(\pi)$ is a tree and $g_j = 0$ for all $j \in \mathcal{J}$. In this article we will assume that M, or equivalently the corresponding plumbing graph, satisfies this additional property.

We recall also that M is an *integral homology sphere*, i.e. $H_1(M, \mathbb{Z}) = 0$, if and only if additionally I is unimodular, i.e. $\det(I) = \pm 1$.

2.1.4. As we already said, by the plumbing construction, any resolution graph $\Gamma(\pi)$ determines the oriented 3-manifold M completely. The converse is also true in the following sense. We say that two graphs (with negative definite intersection forms) are equivalent if one of them can be obtained from the other by a finite sequence of blow-ups and/or blow-downs along rational (-1)-curves. Obviously, for a given $(X, 0)$, the resolution π, hence the graph $\Gamma(\pi)$ too, is not unique. But different resolutions provide equivalent graphs. By a result of W. Neumann [41], the oriented diffeomorphism type of M determines completely the equivalence class of $\Gamma(\pi)$.

2.1.5. In the sequel Γ will denote either a good resolution graph, or a plumbing graph of M. Moreover, $\tilde{X}$ denotes either the space of a good resolution, or the oriented 4-manifold obtained by plumbing disc-bundles corresponding to a plumbing graph.

2.2 The combinatorics of the link.

By the 'combinatorics of the link' we understand the combinatorial machinery related with a fixed resolution (plumbing) graph, or with different lattices associated with it.

2.2.1. Definition. The lattices L and L'. The exact sequence of $\mathbb{Z}$-modules

$$0 \to L \xrightarrow{i} L' \to H \to 0 \tag{1}$$

will stand for the homological exact sequence

$$0 \to H_2(\tilde{X}, \mathbb{Z}) \to H_2(\tilde{X}, M, \mathbb{Z}) \xrightarrow{\partial} H_1(M, \mathbb{Z}) \to 0, \tag{2}$$

or, via Poincaré duality, for

$$0 \to H^2_c(\tilde{X}, \mathbb{Z}) \to H^2(\tilde{X}, \mathbb{Z}) \to H^2(M, \mathbb{Z}) \to 0. \tag{3}$$

Hence, L, considered as in (2), is freely generated by the homology classes $\{E_j\}_{j\in\mathcal{J}}$ (we prefer to use the same notation for the exceptional curves, their homology classes, and the bases of the lattice). For each j, consider a small transversal disc D_j in $\tilde{X}$ with $\partial D_j \subset \partial\tilde{X}$. Then L' is freely generated by the (relative homology) classes $\{D_j\}_{j\in\mathcal{J}}$. Notice that the morphism $i : L \to L'$ can be identified with $L \to \mathrm{Hom}(L, \mathbb{Z})$ given by $l \mapsto (l, \cdot)$. The intersection form has a natural extension to $L_{\mathbb{Q}} = L \otimes \mathbb{Q}$, and it is convenient to identify L' with a sub-lattice of $L_{\mathbb{Q}}$: $\alpha \in \mathrm{Hom}(L, \mathbb{Z})$ corresponds with the unique $l_\alpha \in L_{\mathbb{Q}}$ which satisfies $\alpha(l) = (l_\alpha, l)$ for any $l \in L$. By this identification, D_j, considered in $L_{\mathbb{Q}}$ (and written in the base $\{E_j\}_j$), is the j^{th} column of I^{-1}, and $(D_j, E_i) = \delta_{ji}$.

This shows that the exact sequence (1) can be deduced from $\Gamma(\pi)$ *– or, in view of 2.1.4, from* M *– i.e.* L *is the free* $\mathbb{Z}$*-module generated by the vertices of* $\Gamma(\pi)$*, the bilinear form is* I*, while* L' *is the dual of* $(L,(\ ,\ ))$.

2.2.2. Elements $x=\sum_j r_jE_j\in L_{\mathbb{Q}}$ will be called (rational) cycles. If $x_i=\sum_j r_{j,i}E_j$ for $i=1,2$, then $\min\{x_1,x_2\}:=\sum_j \min\{r_{j,1},r_{j,2}\}E_j$. x^2 means (x,x). We define the support $|x|$ of x by $\cup E_j$, where the union runs over $\{j: r_j\neq 0\}$.

2.2.3. 'Positive' cones. One can consider two types of 'positivity conditions' for rational cycles. The first one is considered in L. A cycle $x=\sum_j r_jE_j\in L_{\mathbb{Q}}$ is called *effective*, denoted by $x\geq 0$, if $r_j\geq 0$ for all j. Their collection is denoted by $L_{\mathbb{Q},e}$, while $L'_e:=L_{\mathbb{Q},e}\cap L'$ and $L_e:=L_{\mathbb{Q},e}\cap L$. We write $x\geq y$ if $x-y\geq 0$. $x>0$ means $x\geq 0$ but $x\neq 0$. $\geq$ provides a partial order of $L_{\mathbb{Q}}$.

The second is the *numerical effectiveness* of the rational cycles, i.e. positivity considered in L'. We define $L_{\mathbb{Q},ne}:=\{x\in L_{\mathbb{Q}}:(x,E_j)\geq 0 \text{ for all } j\}$. In fact, $L_{\mathbb{Q},ne}$ is the positive cone in $L_{\mathbb{Q}}$ generated by $\{D_j\}_j$, i.e. it is exactly $\{\sum_j r_jD_j, r_j\geq 0 \text{ for all } j\}$. Since I is negative definite, all the entries of D_j are *strictly* negative. In particular, $-L_{\mathbb{Q},ne}\subset L_{\mathbb{Q},e}$. Similarly as above, write $L_{ne}:=L\cap L_{\mathbb{Q},ne}$.

Later, in 2.2.7, we will discuss a series of crucial elements of these cones. The cycle which stands as a model for all of them is the classical Artin's cycle, introduced in [3, 4]. Artin proved that, if x_1,x_2 are elements of $-L_{ne}\setminus\{0\}$, then $\min\{x_1,x_2\}$ is also an element of $-L_{ne}\setminus\{0\}$. In particular, $-L_{ne}\setminus\{0\}$ has a unique minimal element, Z_{min}, the *Artin's fundamental (or minimal) cycle.*

2.2.4. The canonical cycle. The canonical divisor $K_{\tilde{X}}$ of $\tilde{X}$ numerically is codified by the *canonical cycle* $K\in L'$, defined by the system (of adjunction relations) $(K,E_j)=-(E_j,E_j)-2$ for all j. Since I is non-degenerate, the system has a unique solution in L'.

Although the self–intersection K^2 depends on the choice of the resolution π, the rational number $K^2+\#\mathcal{J}$ is independent of the choice of π, and it is an invariant of the link M.

Another crucial importance of the canonical cycle K consists of its role in the Riemann-Roch formula. For this, we fix an integral cycle $x\in L_e\setminus\{0\}$. Although the individual cohomology dimensions $h^i(x):=h^i(\tilde{X},\mathcal{O}_x)$ $(i=1,2)$, in general, are not topological, the Euler characteristic $\chi(x):=h^0(x)-h^1(x)$ depends only on $\Gamma(\pi)$ by the Riemann-Roch theorem: $\chi(x)=-(x,x+K)/2$.

2.2.5. Definitions. We say that $(X,0)$ is *numerically Gorenstein* if K has integral coefficients, i.e., if $K\in L$. (This is the 'topological counterpart' of Gorenstein property. Indeed, $(X,0)$ is said to be Gorenstein if the line bundle $\Omega^2_{X\setminus\{0\}}$ is holomorphically trivial. The topological analogue of this fact — which happens if and only if $K\in L$ — is when this line bundle is topologically trivial.)

2.2.6. Characteristic elements. ***Spin***^{c}**-structures.** The set of characteristic elements are defined by

$$Char=Char(L):=\{k\in L':\ (k,x)+(x,x)\in 2\mathbb{Z} \text{ for any } x\in L\}.$$

Notice that the canonical cycle (2.2.4) is in $Char$ and $Char=K+2L'$. There is a natural action of L on $Char$ by $x*k:=k+2x$ whose orbits are of type $k+2L$. Obviously, H acts freely and transitively on the set of orbits by $[l']*(k+2L):=k+2l'+2L$ (in particular, they have the same cardinality).

If $\tilde{X}$ is a resolution as above, then the first Chern class (of the associated determinant line bundle) realizes an identification between the *spin*^{c}-structures $Spin^c(\tilde{X})$ on $\tilde{X}$ and $Char\subset L'=H^2(\tilde{X},\mathbb{Z})$ (see e.g. [11], 2.4.16). The restrictions to M defines an identification of the *spin*^{c}-structures $Spin^c(M)$ of M with the set of orbits of $Char$ modulo $2L$; and this identification is compatible with the action of H on both sets. In the sequel, we think about $Spin^c(M)$ by this identification, hence any *spin*^{c}-structure of M will be represented by an orbit $[k]:=k+2L\subset Char$. The *canonical spin*^{c} structure corresponds to $[K]$.

2.2.7. Liftings. If H is not trivial, then the exact sequence (2.2.1)(1) does not split. Nevertheless, we will consider some 'liftings' (set theoretical sections) of the element of H into L'. They correspond to the positive cones in $L_{\mathbb{Q}}$ considered in 2.2.3.

More precisely, for any $l'+L=h\in H$, let $l'_e(h)\in L'$ be the unique minimal effective rational cycle in $L_{\mathbb{Q},e}$ whose class is h. Clearly, the set $\{l'_e(h)\}_{h\in H}$ is exactly $Q:=\{\sum_j r_jE_j\in L'\,;\,0\leq r_j<1\}$ (the intersection of L' with the closed/open unit rational 'L-cube').

Similarly, for any $h=l'+L$, the intersection $(l'+L)\cap L_{\mathbb{Q},ne}$ has a unique maximal element $l'_{ne}(h)$, and the intersection $(l'+L)\cap(-L_{\mathbb{Q},ne}))$ has a unique minimal element $\bar{l}'_{ne}(h)$ (cf. [34], 5.4). By their definitions $\bar{l}'_{ne}(h)=-l'_{ne}(-h)$.

The elements $\bar{l}'_{ne}(h)$ were introduced in [34] and were denoted there by $l'_{[k]}$, where $[k]=K+2(l'+L)$. Using these elements, one defines the distinguished representative k_r of $[k]$ by $k_r:=K+2\bar{l}'_{ne}(h)$ ($h=l'+L$). Sometimes, in the body of the paper, we will use these notations as well.

For some h, $\bar{l}'_{ne}(h)$ might be situated in Q, but, in general, this is not the case (cf. 4.5.3 and 4.5.4). In general, the characterization of all the elements $\bar{l}'_{ne}(h)$ is not simple (see 3.6.2 when M is a lens space, or [34] for Seifert manifold).

2.2.8. The χ-functions (Riemann-Roch formula). For any characteristic element $k\in Char$ one defines

$$\chi_k: L'\to\mathbb{Q}\ \text{ by }\ \chi_k(l'):=-(l',l'+k)/2.$$

Clearly, $\chi_k(L)\subset\mathbb{Z}$. For $k=K$ we recover the classical Riemann-Roch function (cf. 2.2.4). For the interpretation of χ_k in terms of (twisted) Riemann-Roch, consider the following.

Fix a line bundle $\mathcal{L}\in Pic(\tilde{X})$, and set $c_1(\mathcal{L})=l'\in L'$ (cf. 4.2). Set $k:=K-2l'\in Char$. For any $l\in L$ with $l>0$ one defines the sheaf $\mathcal{O}_l:=\mathcal{O}_{\tilde{X}}/\mathcal{O}_{\tilde{X}}(-l)$ supported by E. Consider the sheaf $\mathcal{L}\otimes\mathcal{O}_l$ and let $\chi(\mathcal{L}\otimes\mathcal{O}_l)=h^0(\mathcal{L}\otimes\mathcal{O}_l)-h^1(\mathcal{L}\otimes\mathcal{O}_l)$ be its (holomorphic) Euler-characteristic. The Riemann-Roch theorem states that this can be computed combinatorially, namely

$$\chi(\mathcal{L}\otimes\mathcal{O}_l)=-(l,l+k)/2=\chi_k(l).$$

2.3 The topology of the link. The Heegaard Floer homology $HF^+(-M)$.

2.3.1. The list of invariants of connected (oriented) 3–manifolds is huge. Any of them can be considered as a topological invariant of the singularity. The invariant becomes really interesting (for algebraic geometers) if it can be related with the analytic structure of the germ.

This list is separated into two, rather distinguishable parts. Some invariants are motivated by singularity theory, are described e.g. in terms of the lattice L, or their positive cones. The other part is produced by the classical 3–manifold invariants developed by topologists. Sometimes it is difficult to find a dictionary connecting them.

For example, our (negative definite plumbed) three manifolds are completely characterized by their fundamental groups (here, in the $\mathbb{Q}HS$ *case, the lens spaces are exceptions, but they are also well understood). Then, one might ask, how can one understand the integer* Z^2_{min} *(associated, say, with the minimal good resolution) in terms of* $\pi_1(M)$*?*

In this article we will concentrate mainly on a rather new invariant, the Seiberg-Witten invariant of M*. There are many ways to introduce it (here we will mention briefly three of them). One of them it realizes as the 'euler-characteristic' of Heegaard–Floer homology.*

2.3.2. The Ozsváth–Szabó invariant. For any oriented rational homology 3-sphere M the Heegaard Floer homology $HF^+(M)$ was introduced by Ozsváth and Szabó in [50] (see also their long list of articles). In fact, $HF^+(M)$ is a $\mathbb{Z}[U]$-module with a $\mathbb{Q}$-grading compatible with the $\mathbb{Z}[U]$-action, where $\deg(U)=-2$. Additionally, $HF^+(M)$ also has an (absolute) $\mathbb{Z}_2$-grading; $HF^+_{even}(M)$, respectively $HF^+_{odd}(M)$, denote the part of $HF^+(M)$ with the corresponding parity.

Moreover, $HF^+(M)$ has a natural direct sum decomposition of $\mathbb{Z}[U]$-modules (compatible with all the gradings) corresponding to the *spin*^{c}-structures of M:

$$HF^+(M)=\oplus_{\sigma\in Spin^c(M)}\,HF^+(M,\sigma).$$

For any *spin*^{c}-structure σ, one has a graded $\mathbb{Z}[U]$-module isomorphism

$$HF^+(M,\sigma)=\mathcal{T}^+_{d(M,\sigma)}\oplus HF^+_{red}(M,\sigma),$$

where $HF^+_{red}(M,\sigma)$ has a finite $\mathbb{Z}$-rank and an induced (absolute) $\mathbb{Z}_2$-grading, $\mathcal{T}^+_d$ is an irreducible $\mathbb{Z}[U]$–module of infinite $\mathbb{Z}$–rank such that $\ker(U|\mathcal{T}^+_d)$ has rank one of degree d (for a more precise definition, see 3.2.1); in fact, $d(M,\sigma)$ can also be defined via this isomorphism. One also considers

$$\chi(HF^+(M,\sigma)) := \mathrm{rank}_{\mathbb{Z}}\, HF^+_{red,even}(M,\sigma) - \mathrm{rank}_{\mathbb{Z}}\, HF^+_{red,odd}(M,\sigma).$$

Then one recovers the Seiberg-Witten topological invariant of (M,σ) (see [56]) via

$$\mathbf{sw}^{OSz}(M,\sigma) := \chi(HF^+(M,\sigma)) - \frac{d(M,\sigma)}{2}.$$

With respect to the change of orientation the above invariants behave as follows: The *spin*^{c}-structures $Spin^c(M)$ and $Spin^c(-M)$ are canonically identified (where $-M$ denotes M with the opposite orientation). Moreover,

$$d(M,\sigma) = -d(-M,\sigma) \quad \text{and} \quad \chi(HF^+(M,\sigma)) = -\chi(HF^+(-M,\sigma)).$$

Notice also that one can recover $HF^+(M,\sigma)$ from $HF^+(-M,\sigma)$ via (7.3) [50] and (1.1) [52].

2.3.3. We will use the notation $\lambda(M)$ for the **Casson-Walker invariant of** M (normalized as in [23] (4.7)). If M is an integral homology sphere (i.e., if H is trivial), then there is only one *spin*^{c} structure, the canonical one, σ_{can}, and by [49] (1.3):

$$\lambda(M) = \mathbf{sw}^{OSz}(M,\sigma_{can}).$$

2.3.4. The 'original' definition of the 'topological' Seiberg-Witten invariant is the following.

For any fixed *spin*^{c}-structure $\sigma \in Spin^c(M)$, one defines the *modified Seiberg-Witten invariant* $\mathbf{sw}^0_M(\sigma)$ as the sum of the number of Seiberg-Witten monopoles and the Kreck-Stolz invariant, see [6, 24, 27] (for this notation, more discussions and references, and relevance with singularities, see [37]). In the present article we prefer to change its sign: we will write $\mathbf{sw}^0(M,\sigma) := -\mathbf{sw}^0_M(\sigma)$. In general it is very difficult to compute $\mathbf{sw}^0(M,\sigma)$ using its analytic definition, therefore there is an intense activity to replace this definition with a different one. Besides the Ozsváth–Szabó theory, this invariant can also be recovered by the *sign refined Reidemeister-Turaev torsion* $\mathcal{T}_{M,\sigma} = \sum_{h\in H} \mathcal{T}_{M,\sigma}(h)h \in \mathbb{Q}[H]$ (determined by the Euler structure) associated with σ [62] via

$$\mathbf{sw}^{TCW}(M,\sigma) := -\mathcal{T}_{M,\sigma}(1) + \lambda(M)/|H|.$$

Here 1 denotes the neutral element of the group H (with the multiplicative notation).

For negative definite plumbed 3–manifolds one has ([56, 47]): $\mathbf{sw}^0(M,\sigma) = \mathbf{sw}^{TCW}(M,\sigma) = \mathbf{sw}^{OSz}(M,\sigma)$. Nevertheless, different realizations might illuminate essentially different aspects of the theory. E.g., since $\sum_\sigma \mathcal{T}_{M,\sigma}(1) = 0$, for any rational homology sphere one has

$$\lambda(M) = \sum_\sigma \mathbf{sw}^{TCW}(M,\sigma). \tag{1}$$

If we do not want to specify the source of the invariant, we just write $\mathbf{sw}(M,\sigma)$.

2.4 Some analytic invariants of the singularity.

2.4.1. Definitions. The analytic type of $(X,0)$ is characterized completely by its local analytic ring $\mathcal{O}_{X,0}$ whose maximal ideal will be denoted by $m_0 \subset \mathcal{O}_{X,0}$. Here are some of its discrete invariants.

- The *Hilbert-Samuel function* is defined by

$$f_{HS}(k) = \dim_{\mathbb{C}} \mathcal{O}_{X,0}/m_0^k \quad \text{for any } k \geq 1.$$

Then $f_{HS}(1) = 1$ and $f_{HS}(2) - 1 = \dim m_0/m_0^2$ equals the minimal N for which some embedding $(X,0) \subset (\mathbb{C}^N,0)$ can be realized, hence is called the *embedding dimension* of $(X,0)$. For $k \gg 1$, $f_{HS}(k) = P_{HS}(k)$ for some polynomial P_{HS} (called the Hilbert-Samuel polynomial)

$$P_{HS}(k) = mk^2/2 + a_1k + a_2.$$

The integer m above is called the *multiplicity* of $(X,0)$, and it is denoted by $mult(X,0)$. It is not difficult to verify that if $(X,0)\subset(\mathbb{C}^N,0)$ is an arbitrary embedding and ℓ a generic affine space of codimension 2 (close to the origin), then $mult(X,0)=\#X\cap\ell$.

• The *geometric genus* is defined by $p_g:=\dim_{\mathbb{C}} H^1(\tilde{X},\mathcal{O}_{\tilde{X}})$, where $\tilde{X}\to X$ is any resolution as above. It can be recovered from the sheaf-cohomology of some one-dimensional spaces as well: by the theorem of formal functions, for $x\in L$, $x=\sum_{j\in\mathcal{J}} m_jE_j$ with $m_j\gg 0$, one has $p_g=h^1(x)$.

For a generalization of p_g, where the structure sheaf $\mathcal{O}_{\tilde{X}}$ will be replaced by line bundles on $\tilde{X}$, see § 4.

2.4.2. Question: Are the analytic invariants topological? In the theory of surface singularities, one of the guiding questions is the following: *is it possible to recover (some of) the analytic invariants from the link M, or equivalently, from a resolution graph $\Gamma(\pi)$ of $(X,0)$?* Even more, *is it possible to characterize some important families of singularities (defined a priori via analytic terms) by their topology.* For a rather detailed discussion of this problem see the review article [33]. But, in some sense, the whole philosophy of the present article, and some of its main results, are also motivated by these questions.

The 'classical' models for such invariant and family–characterizations are the cyclic quotient, rational and (weakly) elliptic singularities. For a detailed discussion of their properties, see also [30, 33], and references therein.

2.4.3. Example. Hirzebruch-Jung singularities, by definition, are characterized by the existence of a finite projection $(X,0)\to(\mathbb{C}^2,0)$ whose reduced discriminant space is included in the union of the coordinate axes of $(\mathbb{C}^2,0)$. On the other hand, one can show that $(X,0)$ is Hirzebruch-Jung if and only if its link is a lens space $L(p,q)$ with $0<q<p$ and $gcd(p,q)=1$; or equivalently, if the minimal resolution graph is a straight line graph with all genera zero:

where $-k_1,\dots,-k_s$ are given by the continued fraction $[k_1,k_2,\dots,k_s]$:

$$p/q=[k_1,k_2,\dots,k_s]=k_1-\cfrac{1}{k_2-\cfrac{1}{\ddots-\cfrac{1}{k_s}}},\quad k_1,\dots,k_s\geq 2.$$

Notice also that a Hirzebruch-Jung singularity can also be realized as a *cyclic quotient* singularity $X_{p,q}:=(\mathbb{C}^2,0)/\mathbb{Z}_p$. Here the action is $\xi*(u,v)=(\xi u,\xi^q v)$, where ξ is a primitive p-th root of unity.

2.4.4. Example. Rational singularities were defined by Artin by the vanishing of the geometric genus $p_g=0$. The next subsection (2.5) is devoted to their topological characterization. Subsection (2.6) recalls the relevant facts about the **elliptic singularities**.

2.5 Rational singularities.

2.5.1. Recall that $(X,0)$ is rational if $p_g=0$. In the sequel we fix a resolution π of $(X,0)$. It is easy to see that $p_g=0$ if and only if $h^1(x)=0$ for any $x\in L$, $x>0$. (In particular, all the genera g_j should vanish, and $\Gamma(\pi)$ should be a tree.)

The main point is that Artin succeeded in replacing the vanishing of $h^1(x)$'s by a criterion formulated in terms of χ. In fact, it is enough to consider only one cycle, namely the fundamental cycle Z_{min}. It is instructive to recall that for *any* normal surface singularity $h^0(Z_{min})=1$, hence $\chi(Z_{min})\leq 1$.

2.5.2. Topological characterizations of rational singularities [3, 4].

$$p_g=0 \quad\Leftrightarrow\quad \chi(x)\geq 1 \ \text{ for all cycles } x>0 \quad\Leftrightarrow\quad \chi(Z_{min})=1.$$

Notice that these characterizations are independent of the choice of the resolution π. If a resolution graph satisfies the second (or equivalently, the third) property of the above equivalence, we say that it is a *rational graph*. Any rational graph is automatically good.

2.5.3. Examples. (a) The *rational double points (RDP)*, (i.e. rational singularities with multiplicity two) are exactly the (simple) hypersurface singularities of type A-D-E. Topologically they are characterized by the fact that their minimal resolution graphs are the well–known A-D-E (negative definite) graphs. In fact, any connected, negative definite graph with $g_j = e_j + 2 = 0$ for all j is the minimal resolution graph of some RDP (and it is of type A-D-E).

(b) Let Γ be an arbitrary tree. For any vertex j set δ_j to be the number of edges with endpoint j. Consider the decorations $g_j = 0$ and $e_j = \begin{cases} -\delta_j & \text{if } \delta_j \neq 1 \\ -2 & \text{if } \delta_j = 1 \end{cases}$ for any j. Then the intersection matrix I is automatically negative definite, hence Γ is the minimal resolution graph of some singularity. One can show that $Z_{min} = \sum_j E_j$, $\chi(Z_{min}) = 1$, hence *any* singularity with minimal resolution graph Γ is rational. Moreover, $Z_{min}^2 = -\#\{v : \delta_j = 1\}$. (This also shows that the multiplicity and the embedded dimension of a rational singularity can be arbitrarily large, cf. 2.5.4.)

(c) The class of *rational graphs* is closed while taking subgraphs and decreasing self-intersections. In particular, in the above example (b), one can decrease any of the decorations e_j, and still obtain a rational graph. For these modified graphs one still has $Z_{min} = \sum_j E_j$. Rational surface singularities with reduced fundamental cycle (i.e. with $Z_{min} = \sum E_j$) are also called *minimal singularities*. E.g., the cyclic quotients are minimal.

(d) Another subclass of rational singularities has the name of *sandwiched singularities*. A sandwiched singularity is, by definition, a normal surface singularity that admits a birational map to $(\mathbb{C}^2, 0)$. If we consider a resolution $\tilde{X} \to X$, then we get a diagram $(\tilde{X}, E) \to (X, 0) \to (\mathbb{C}^2, 0)$, hence X is sandwiched between two smooth spaces via birational maps.

They also can be characterized by their (minimal) resolution graphs as follows [57]. Consider a plane curve singularity $(C, 0) \subset (\mathbb{C}^2, 0)$, and let $Y \to \mathbb{C}^2$ be a (in general, non-minimal) embedded resolution ϕ of it. Consider the collection E of those irreducible exceptional divisors which are not (-1)-curves (and assume that they form a connected curve). If one contracts E then one gets a sandwiched singularity, and any of them can be obtained in this way (although the choice of $(C, 0)$ and ϕ is not unique). Notice that this can be reformulated in terms of the combinatorics of the graph as well.

The next result targets the analytic invariants introduced above.

2.5.4. Theorem. [3, 4] *Assume that $(X, 0)$ is rational and $k \geq 1$. Then $f_{HS}(k) = -k(k-1)/2 \cdot Z_{min}^2 + k$. In particular, $mult(X, 0) = -Z_{min}^2$ and $emb\,dim(X, 0) = -Z_{min}^2 + 1$.*

2.6 Weakly elliptic singularities.

2.6.1. Definition. A normal surface singularity $(X, 0)$ is called *weakly elliptic*, in short *elliptic*, if its graph $\Gamma(\pi)$ is elliptic. A graph $\Gamma(\pi)$ is elliptic if $\min_{x>0} \chi(x) = 0$, or equivalently, $\chi(Z_{min}) = 0$. (Wagreich used the first vanishing [63], Laufer the second one [22]; see also [30] for their equivalence.) The definition is independent of the choice of the resolution graph.

2.6.2. Remark. The set of elliptic singularities includes all the singularities with $p_g = 1$, and all the Gorenstein singularities with $p_g = 2$. But an elliptic singularity might have arbitrary large p_g.

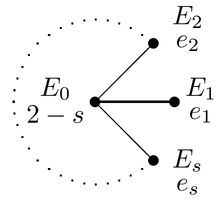
In general, the geometric genus and the other analytic invariants listed in 2.4.1 of an elliptic singularity are not topological (for discussion, see e.g. [30, 33]). But Laufer, in [22], identified topologically a subclass of elliptic singularities for which p_g is topological (in fact, $p_g = 1$). For this characterization, it is convenient to consider the minimal resolution of $(X, 0)$ (i.e. a resolution which has no rational (-1) curve).

2.6.3. Theorem. Minimally elliptic singularities. [22] *Consider the minimal resolution π of $(X, 0)$. Then the following statements are equivalent. If a singularity satisfies them, it is called minimally elliptic.*

(i) $(X,0)$ *is numerically Gorenstein and* $-K = Z_{min}$.
(ii) $\chi(Z_{min}) = 0$ *and any proper subgraph of* $\Gamma(\pi)$ *supports a rational singularity.*
(iii) $p_g = 1$ *and* $(X,0)$ *is Gorenstein.*

Notice that (i-ii) give topological characterizations of Gorenstein singularities with $p_g = 1$. For additional characterizations, see [22], or [30]. Here is an example when the link is $\mathbb{Q}HS$:

2.6.4. Example. 'Polygonal' singularities. Assume that the resolution graph Γ has the following form with $s+1$ vertices (with $g_j = 0$):



The intersection matrix is negative definite if and only if $2 - s + \sum_{j>0} 1/e_j < 0$.

If $s > 3$ then Γ is minimal, and $-K = Z_{min} = 2E_0 + \sum_{j>0} E_j$.

If $s = 3$, then $Z_{min} \neq -K$. But in this case Γ is not minimal: E_0 is a rational (-1)-curve, so it should be contracted. After blow down, in the minimal graph, $-K = Z_{min} = \sum E_j$ again. If $s = 3$ then $(X,0)$ is the *Dolgachev's triangle singularity* D_{e_1-1,e_2-1,e_3-1}.

2.6.5. Remark. Laufer in [22] computed the Hilbert-Samuel function of minimally elliptic singularities from their graphs. This is generalized by the author in [32] *for Gorenstein singularities with* $b_1(M) = 0$. In this description, the crucial topological ingredient is the *elliptic sequence of an elliptic singularity*, introduced and intensively studied by S.S.-T. Yau, see [65, 66]. We prefer to recall it only in the numerically Gorenstein case. We consider a *minimal* resolution π with $E = \pi^{-1}(0)$, and we write Z_K for $-K$. In such a case $Z_K \geq Z_{min}$.

2.6.6. Definition. The elliptic sequence consists of a sequence $\{Z_{B_j}\}_{j=0}^m$, where Z_{B_j} is the fundamental cycle of $B_j \subset E$. We define $\{B_j\}_j$ inductively as follows. For $j = 0$ take $B_0 = E$, hence $Z_{B_0} = Z_{min}$. Then $Z_K \geq Z_{B_0}$. If $Z_K > Z_{B_0}$ then we set $B_1 := |Z_K - Z_{B_0}|$. Similarly, if B_i is already defined for any $i \leq j$, then it turns out that $Z_K \geq Z_{B_0} + \cdots + Z_{B_j}$. If the inequality is strict then we define $B_{j+1} := |Z_K - Z_{B_0} - \cdots - Z_{B_j}|$, otherwise we stop. In particular, $Z_K = \sum_{j=0}^m Z_{B_j}$. The length of the elliptic sequence $\{Z_{B_j}\}_{j=0}^m$ is $m+1$. The case $m = 0$ (i.e. when the identity $Z_K = Z_{min}$ holds) corresponds exactly to the minimally elliptic singularities.

It is worth mentioning that by Yau's inequality (cf. [66, (3.9)]), $m+1$ is an optimal topological upper bound for p_g: for a numerically Gorenstein elliptic singularity $p_g \leq m+1$. This (and other partial results of Yau and Tomari [60]) were the starting points of the following result:

2.6.7. Theorem.[32] *Assume that the link of the elliptic Gorenstein singularity* $(X,0)$ *is a rational homology sphere. Then* p_g *is a topological invariant: it equals the length of the elliptic sequence in the minimal resolution of* $(X,0)$. *Moreover, the following holds:*

(a) $mult(X,0) = \max(2, -Z_{min}^2)$;
(b) $emb\ dim(X,0) = \max(3, -Z_{min}^2)$;
(c) If $Z_{min}^2 \leq -3$ *then* $\dim m_0^k/m_0^{k+1} = -kZ_{min}^2$ $(k \geq 1)$.

2.7 Almost rational singularities.

Recall that rational singularities can be characterized by their graphs; those graphs which satisfy the corresponding combinatorial properties are called *rational graphs*, cf. 2.5.2. The next definition is in the same spirit: using a combinatorial definition we enlarge the set of rational (plumbing) graphs:

2.7.1. Definition. Assume that the plumbing graph Γ is a negative definite connected tree. We say that Γ is *almost-rational* (in short AR) if there exists a vertex $j_0 \in \mathcal{J}$ of Γ such that replacing e_{j_0} by some $e'_{j_0} \leq e_{j_0}$ we get a rational graph Γ'. In general, the choice of j_0 is not unique. Once the distinguished vertex j_0 is fixed, we write $\mathcal{J} = \{0\} \cup \mathcal{J}^*$ such that the index $0 \in \mathcal{J}$ corresponds to this vertex.

2.7.2. Examples. The set of AR graphs is surprisingly large.

1) Obviously, all rational graphs are AR.

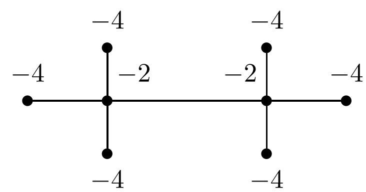
2) Any elliptic graph is AR (for a proof, see [34], (8.2)).

3) Any star-shaped graph is AR. Indeed, first blow down all the (-1)-vertices different from the central vertex (this transformation preserves the AR graphs); let this new graph be $\bar{\Gamma}$. Then take for j_0 the central vertex of $\bar{\Gamma}$, and take for $-e'_{j_0}$ an integer larger than the number of adjacent vertices of the central vertex of $\bar{\Gamma}$. Then the modified graph will become minimal rational, cf. 2.5.3. (In other words: all the Seifert 3–manifolds are plumbed manifolds associated with AR graphs.)

4) The class of AR graphs is closed while taking subgraphs and decreasing the Euler numbers e_j (since the class of rational graphs is so).

5) The rational surgery 3–manifolds $S^3_{-p/q}(K)$ considered in §5 are AR (see 5.2.4).

6) Not any graph is AR. For example, if Γ has two (or more) vertices, both with decoration $-e_j \leq \delta_j - 2$, then Γ is not AR. E.g.:



This graph has the following property too: if we delete one of the (-2)–vertices, then all the components of the remaining graph are rational. Still the graph itself is not AR.

2.7.3. In fact, all the main results of the present article are related with AR graphs. Their generalization for the general case looks rather difficult (and it can be a beautiful goal for interested reader).

3 Graded roots [34]

3.1 Graded roots.

3.1.1. Preliminary remarks. The Ozsváth-Szabó $\mathbb{Z}[U]$-module $HF^+(M, [k])$ can be computed (for any plumbed 3–manifolds M associated with an AR graph, see below) in a combinatorial way from the corresponding plumbing graph Γ. Our goal is to define an intermediate object, a graded root R_k associated with *any* negative definite plumbed graph Γ and a characteristic element k. This will contain all the needed information to determine the homological object HF^+, but it preserves also some additional, more subtle topological information about Γ (or, about M).

In this and next subsection we give the definition and first properties of abstract graded roots. Subsection 3.3 contains the construction of the graded roots R_k from the plumbing graphs Γ. [Although both Γ (the plumbing graph) and the constructed graded root R_k are "connected trees", they serve rather different roles. E.g., the edges of Γ codify the corresponding gluings in the plumbing, while the edges of R_k codify the $\mathbb{Z}[U]$-action. We hope the terminology will not create any confusion.]

3.1.2. Definitions.

(1) Let R be an infinite tree with vertices $\mathcal{V}$ and edges $\mathcal{E}$. We denote by $[u, v]$ the edge with end-points u and v. We say that R is a *graded root* with grading $\chi : \mathcal{V} \to \mathbb{Z}$ if

(a) $\chi(u) - \chi(v) = \pm 1$ for any $[u, v] \in \mathcal{E}$;

(b) $\chi(u) > \min\{\chi(v), \chi(w)\}$ for any $[u, v]$, $[u, w] \in \mathcal{E}$, $v \neq w$;

(c) χ is bounded below, $\chi^{-1}(k)$ is finite for any $k \in \mathbb{Z}$, and $\#\chi^{-1}(k) = 1$ if k is sufficiently large.

(2) $v \in \mathcal{V}$ is a *local minimum point* of the graded root (R, χ) if $\chi(v) < \chi(w)$ for any edge $[v, w]$. Their set is denoted by $\mathcal{V}_{lm}$. In fact, $\mathcal{V}_{lm}$ coincides with the set of vertices with adjacent degree one.

(3) A geodesic path connecting two vertices is *monotone* if χ restricted to the set of vertices on the path is strict monotone. If a vertex v can be connected by another vertex w by a monotone geodesic and $\chi(v) > \chi(w)$, then we write $v \succ w$. $\succ$ is an ordering of $\mathcal{V}$. For any pair $v, w \in \mathcal{V}$ there is a unique $\succ$-minimal vertex $\sup(v, w)$ which dominates both.

(4) If (R, χ) is a graded root, and $r \in \mathbb{Z}$, then we denote by $(R, \chi)[r]$ the same R with the new grading $\chi[r](v) := \chi(v) + r$. (This can be generalized for any $r \in \mathbb{Q}$ as well.)

3.1.3. Examples. (1) For any integer $n \in \mathbb{Z}$, let R_n be the tree with $\mathcal{V} = \{v^k\}_{k \geq n}$ and $\mathcal{E} = \{[v^k, v^{k+1}]\}_{k \geq n}$. The grading is $\chi(v^k) = k$.

(2) Let I be a finite index set. For each $i \in I$ fix an integer $n_i \in \mathbb{Z}$; and for each pair $i, j \in I$ fix $n_{ij} = n_{ji} \in \mathbb{Z}$ with the next properties: (i) $n_{ii} = n_i$; (ii) $n_{ij} \geq \max\{n_i, n_j\}$; and (iii) $n_{jk} \leq \max\{n_{ij}, n_{ik}\}$ for any $i, j, k \in I$.

For any $i \in I$ consider $R_i := R_{n_i}$ with vertices $\{v_i^k\}$ and edges $\{[v_i^k, v_i^{k+1}]\}$, $(k \geq n_i)$. In the disjoint union $\coprod_i R_i$, for any pair (i, j), identify v_i^k and v_j^k, resp. $[v_i^k, v_i^{k+1}]$ and $[v_j^k, v_j^{k+1}]$, whenever $k \geq n_{ij}$. Write $\bar{v}_i^k$ for the class of v_i^k. Then $\coprod_i R_i/\sim$ is a graded root with $\chi(\bar{v}_i^k) = k$. It will be denoted by $R = R(\{n_i\}, \{n_{ij}\})$.

Clearly $\mathcal{V}_{lm}(R)$ is a subset of $\{\bar{v}_i^{n_i}\}_{i \in I}$, and this last set can be identified with I. $\mathcal{V}_{lm}(R) = I$ if in (ii) all the inequalities are strict. Otherwise all the indices $I \setminus \mathcal{V}_{lm}(R)$ are superfluous, i.e. the corresponding R_i's produce no additional vertices.

In fact, any graded root (R', χ') is isomorphic (in a natural sense) with some $R(\{n_i\}, \{n_{ij}\})$. Indeed, set $I := \mathcal{V}_{lm}(R')$, $n_v := \chi'(v)$ and $n_{uv} := \chi'(\sup(u, v))$ for $u, v \in I$.

(3) Any map $\tau : \{0, 1, \ldots, T_0\} \to \mathbb{Z}$ produces a starting data for construction (2). Indeed, set $I = \{0, \ldots, T_0\}$, $n_i := \tau(i)$ $(i \in I)$, and $n_{ij} := \max\{n_k : i \leq k \leq j\}$ for $i \leq j$. Then $\coprod_i R_i/\sim$ constructed in (2) using this data will be denoted by (R_τ, χ_τ).

For example, for $T_0 = 4$, take for the values of τ: $-3, -1, -2, 0$ and -2 (respectively $-3, 0, -2, -1$ and -2). Then the two graded roots are:

3.2 The homology of a graded root.

3.2.1. $\mathbb{Z}[U]$-modules. We will use the following notations. Consider the graded $\mathbb{Z}[U]$-module $\mathbb{Z}[U, U^{-1}]$, and (following [51]) denote by $\mathcal{T}_0^+$ its quotient by the submodule $U \cdot \mathbb{Z}[U]$. This has a grading in such a way that $\deg(U^{-d}) = 2d$ $(d \geq 0)$. Similarly, for any $n \geq 1$, define the graded module $\mathcal{T}_0(n)$ by the quotient of $\mathbb{Z}\langle U^{-(n-1)}, U^{-(n-2)}, \ldots, 1, U, \ldots\rangle$ by $U \cdot \mathbb{Z}[U]$ (with the same grading). Hence, $\mathcal{T}_0(n)$, as a $\mathbb{Z}$-module, is the free $\mathbb{Z}$-module $\mathbb{Z}\langle 1, U^{-1}, \ldots, U^{-(n-1)}\rangle$ (generated by $1, U^{-1}, \ldots, U^{-(n-1)}$), and has finite $\mathbb{Z}$-rank n.

More generally, for any graded $\mathbb{Z}[U]$-module P with d-homogeneous elements P_d, and for any $r \in \mathbb{Q}$, we denote by $P[r]$ the same module graded in such a way that $P[r]_{d+r} = P_d$. Then set $\mathcal{T}_r^+ := \mathcal{T}_0^+[r]$ and $\mathcal{T}_r(n) := \mathcal{T}_0(n)[r]$.

3.2.2. Definition. The $\mathbb{Z}[U]$-modules associated with graded roots. For any graded root (R, χ), let $\mathbb{H}(R, \chi)$ (briefly $\mathbb{H}(R)$) be the set of functions $\phi : \mathcal{V} \to \mathcal{T}_0^+$ with the following property: whenever $[v, w] \in \mathcal{E}$ with $\chi(v) < \chi(w)$, then

$$U \cdot \phi(v) = \phi(w).$$

Or, equivalently, for any $w \succ v$ one requires

$$U^{\chi(w) - \chi(v)} \cdot \phi(v) = \phi(w). \qquad (*)$$

Clearly $\mathbb{H}(R)$ is a $\mathbb{Z}[U]$-module via $(U\phi)(v) = U \cdot \phi(v)$. Moreover, $\mathbb{H}(R)$ has a grading: $\phi \in \mathbb{H}(R)$ is homogeneous of degree $d \in \mathbb{Z}$ if for each $v \in \mathcal{V}$ with $\phi(v) \neq 0$, $\phi(v) \in \mathcal{T}_0^+$ is homogeneous of degree $d - 2\chi(v)$. Notice that in $(*)$ one has $2\chi(v) + \deg \phi(v) = 2\chi(w) + \deg \phi(w)$, hence d is well-defined.

Notice that any ϕ as above is automatically finitely supported.

3.2.3. From the definitions, it is clear that $\mathbb{H}((R,\chi)[r]) = \mathbb{H}(R,\chi)[2r]$ for any $r \in \mathbb{Z}$.

3.2.4. Proposition. *Let (R,χ) be a graded root. We order $\mathcal{V}_{lm}$ as follows. The first element v_1 is an arbitrary vertex with $\chi(v_1) = \min_v \chi(v)$. If $v_1, \dots, v_k$ is already determined, and $J_k := \{v_1, \dots, v_k\} \subsetneq \mathcal{V}_{lm}$, then let v_{k+1} be an arbitrary vertex in $\mathcal{V}_{lm} \setminus J_k$ with $\chi(v_{k+1}) = \min_{v \in \mathcal{V}_{lm}\setminus J_k} \chi(v)$. Let $w_{k+1} \in \mathcal{V}$ be the unique $\succ$-minimal vertex of R which dominates both v_{k+1}, and at least one vertex from J_k. Then one has the following isomorphism of $\mathbb{Z}[U]$-modules*

$$\mathbb{H}(R,\chi) = \mathcal{T}^+_{2\chi(v_1)} \oplus \oplus_{k\geq 2} \mathcal{T}_{2\chi(v_k)}\big(\chi(w_k) - \chi(v_k)\big).$$

In particular, with the notations $m := \min_v \chi(v)$ and

$$\mathbb{H}_{red}(R,\chi) := \oplus_{k\geq 2} \mathcal{T}_{2\chi(v_k)}\big(\chi(w_k) - \chi(v_k)\big),$$

one has a canonical direct sum decomposition of graded $\mathbb{Z}[U]$-modules: $\mathbb{H}(R,\chi) = \mathcal{T}^+_{2m} \oplus \mathbb{H}_{red}(R,\chi)$.

3.2.5. Examples.(a) $\mathbb{H}(R_n) = \mathcal{T}_{2n}$.

(b) The graded roots R^1 and R^2 constructed in 3.1.3(3) are not isomorphic but their $\mathbb{Z}[U]$-modules are isomorphic: $\mathbb{H}(R^1) = \mathbb{H}(R^2) = \mathcal{T}^+_{-6} \oplus \mathcal{T}_{-4}(1) \oplus \mathcal{T}_{-4}(2)$. Hence, in general, a graded root carries more information than its $\mathbb{Z}[U]$-module.

3.2.6. Corollary. *Let (R_τ, χ_τ) be a graded root associated with some function $\tau : \mathbb{N} \to \mathbb{Z}$, cf. 3.1.3(3), which satisfies $\tau(1) > \tau(0)$. Then the $\mathbb{Z}$-rank of $\mathbb{H}_{red}(R_\tau, \chi_\tau)$ is:*

$$\mathrm{rank}_{\mathbb{Z}}\, \mathbb{H}_{red}(R_\tau) = -\tau(0) + \min_{i\geq 0} \tau(i) + \sum_{i\geq 0} \max\{\tau(i) - \tau(i+1), 0\}.$$

The summand $\mathcal{T}^+_{2m}$ of $\mathbb{H}(T_\tau, \chi_\tau)$ has index $m = \min_{i\geq 0} \tau(i) = \min_v \chi_\tau(v)$.

3.3 Graded roots associated with plumbing graphs.

Fix a connected plumbing graph Γ whose bilinear form is negative definite. In this section we will construct a graded root (R_k, χ_k) associated with any characteristic element k.

3.3.1. The construction of (R_k, χ_k). Fix any $k \in Char$ and define $\chi_k : L \to \mathbb{Z}$ by $\chi_k(x) := -(\,k(x) + (x,x)\,)/2$ as in (2.2.8). For any $n \in \mathbb{Z}$, we define a finite 1-dimensional simplicial complex $\bar{L}_{k,\leq n}$ as follows. Its 0-skeleton is $L_{k,\leq n} := \{x \in L : \chi_k(x) \leq n\}$. For each x and $j \in \mathcal{J}$ with $x, x+E_j \in L_{k,\leq n}$, we consider a unique 1-simplex with endpoints at x and $x+E_j$ (e.g., the segment $[x, x+E_j]$ in $L \otimes \mathbb{R}$). We denote the set of connected components of $\bar{L}_{k,\leq n}$ by $\pi_0(\bar{L}_{k,\leq n})$. For any $v \in \pi_0(\bar{L}_{k,\leq n})$, let C_v be the corresponding connected component of $\bar{L}_{k,\leq n}$.

Next, we define the graded root (R_k, χ_k) as follows. The vertex set $\mathcal{V}(R_k)$ is $\cup_{n\in\mathbb{Z}} \pi_0(\bar{L}_{k,\leq n})$. The grading $\mathcal{V}(R_k) \to \mathbb{Z}$, still denoted by χ_k, is $\chi_k | \pi_0(\bar{L}_{k,\leq n}) = n$.

If $v_n \in \pi_0(\bar{L}_{k,\leq n})$, and $v_{n+1} \in \pi_0(\bar{L}_{k,\leq n+1})$, and $C_{v_n} \subset C_{v_{n+1}}$, then $[v_n, v_{n+1}]$ is an edge of R_k. All the edges $\mathcal{E}(R_k)$ of R_k are obtained in this way.

3.3.2. Proposition. *For any $k \in Char$, (R_k, χ_k) is a graded root.*

3.3.3. Some of these graded roots are not very different. Indeed, assume that k and k' determine the same $spin^c$ structure, hence $k' = k + 2l$ for some $l \in L$. Then $\chi_{k'}(x - l) = \chi_k(x) - \chi_k(l)$ for any $x \in L$. This means that the transformation $x \mapsto x' := x - l$ realizes an identification of $\bar{L}_{k,\leq n}$ and $\bar{L}_{k',\leq n-\chi_k(l)}$. Hence, we get:

$$(R_{k'}, \chi_{k'}) = (R_k, \chi_k)[-\chi_k(l)] \quad \text{whenever } k' = k + 2l \text{ for some } l \in L.$$

In fact, there is an easy way to choose a graded root from the multitude $\{(R_k, \chi_k)\}_{k\in[k]}$. For any $k \in Char$ we define

$$m_k := \frac{k^2 - \max_{k'\in[k]}(k')^2}{8} \leq 0.$$

Since $(k + 2l, k + 2l) = (k, k) - 8\chi_k(l)$, m_k is an integer. Set $M_{[k]} := \{k \in [k] : m_k = 0\}$.

3.3.4. Lemma. *Fix a spin^c structure $[k]$. Then $k_0 \in M_{[k]}$ if and only if $-\chi_{k_0}(l) \leq 0$ for any $l \in L$. Moreover, if k_0 and $k_0+2l \in M_{[k]}$, then $-\chi_{k_0}(l) = 0$. In particular, any choice of $k_0 \in M_{[k]}$ provides the same graded root (R_{k_0}, χ_{k_0}), which will be denoted by $(R_{[k]}, \chi_{[k]})$. Moreover, for any $k \in [k]$*

$$(R_k, \chi_k) = (R_{[k]}, \chi_{[k]})[m_k].$$

The notation is compatible with 3.2.4: $m_k = \min \chi_k$. In fact,

$$k^2 - 8 \min \chi_k = \max_{k'\in[k]}(k')^2.$$

3.3.5. The graded root associated with the *canonical spin^c* structure sometimes is denoted by (R_{can}, χ_{can}). It has the following property: $\#\chi_{can}^{-1}(n) = 1$, provided that $n \geq 1$.

3.3.6. Clearly, many different plumbing graphs can provide the same 3-manifold M. But all these plumbing graphs can be connected by each other by a finite sequence of blowups/downs (-1)-vertices of degree ≤ 2. One can show that $\{(R_{[k]}, \chi_{[k]})\}_{[k]}$ depends only on M, i.e. it is independent of the choice of the (negative definite) plumbing graph Γ which provides M.

3.4 Characterization of rational and elliptic graphs via roots; classification.

We believe that the right object which guides the classification of normal surface singularities is the (canonical) graded root associated with the connected negative definite plumbing graphs. In order to exemplify and support this statement, we start with the following results.

Recall that (R_{can}, χ_{can}) denotes the canonical graded root. We invite the reader to recall the definitions of the roots R_m ($m \in \mathbb{Z}$) in 3.1.3 as well.

3.4.1. Theorem. Characterization of rational graphs [34]. *Let Γ be a connected, negative definite plumbing graph whose plumbed three-manifold is a rational homology sphere. Then the following facts are equivalent:*

(a) Γ *is rational;*
(a′) $\#\chi_{can}^{-1}(0) = 1$;
(b) $R_{can} = R_0$;
(c) $R_{can} = R_m$ *for some* $m \in \mathbb{Z}$;
(d) For all characteristic elements $k \in Char$, $R_k = R_{m_k}$ *for some* $m_k \in \mathbb{Z}$;
(Hb) $\mathbb{H}(R_{can}) = \mathcal{T}_0^+$;
(Hc) $\mathbb{H}(R_{can}) = \mathcal{T}_m^+$ *for some* $m \in \mathbb{Z}$*; or equivalently,* $\mathbb{H}_{red}(R_{can}) = 0$;
(Hd) For all $k \in Char$, $\mathbb{H}(R_k) = \mathcal{T}_{m_k}^+$ *for some* $m_k \in \mathbb{Z}$*; or equivalently,* $\oplus_{[k]}\mathbb{H}_{red}(R_{[k]}) = 0$.

Moreover, if Γ is rational and $k = K + 2l'$, then

$$m_k = \min \chi_k = \min_{x\in L} \chi(l' + x) - \chi(l') = \chi(l'_{[k]}) - \chi(l') = \chi_k(l'_{[k]} - l') \leq 0.$$

In particular, if Γ is rational and $k_r = K + 2l'_{[k]}$, then $\min \chi_{k_r} = 0$.

It is instructive to compare (a') with the property $\#\chi_{can}^{-1}(n) = 1$, valid for any Γ and $n \geq 1$; cf. 3.3.5.

The next result involves only the canonical graded root (the interested reader may clarify the other cases as well):

3.4.2. Theorem. Characterization of elliptic graphs [34]. *Let Γ be a connected, negative definite plumbing graph whose plumbed three-manifold is a rational homology sphere. Then the following facts are equivalent:*

(a) Γ is elliptic;

(b) $R_{can} = R(\{n_i\}, \{n_{ij}\})$ for some index set I, $\#I = l+1 \geq 2$, and $n_i = 0$ for any $i \in I$, and $n_{ij} = 1$ for any $i \neq j$;

(Hb) $\mathbb{H}(R_{can}) = \mathcal{T}_0^+ \oplus \big(\mathcal{T}_0(1)\big)^{\oplus l}$ for some $l \geq 1$.

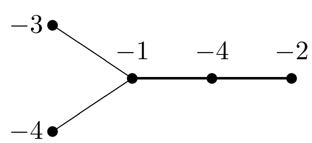
Moreover, l above can be identified with the length of the elliptic sequence. In particular, if the graph Γ is minimally elliptic then $l = 1$.

3.4.3. Remarks. (a) The results 3.4.1 and 3.4.2 can also be interpreted as follows: The grading χ_{can} of R_{can} satisfies $\min \chi_{can} \geq 0$ (or, equivalently $\min \chi_{can} = 0$) if and only if Γ is either rational or weakly elliptic. In this situation, if $\mathbb{H}_{red}(R_{can}) = 0$ then Γ is rational, otherwise it is weakly elliptic. The rank of $\mathbb{H}_{red}(R_{can})$ is the optimal topological upper bound for p_g.

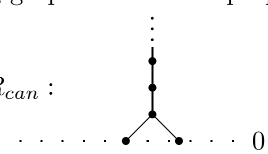
(b) In both rational and weakly elliptic case, for any *spin*^{c}-structure $[k]$, one has $\min \chi_{k_r} = 0$.

(c) One can find elliptic graphs with $l = 1$ which are not numerically Gorenstein (i.e. $K \notin L$), hence which are not minimally elliptic. E.g., the following graph Γ has these properties:

Γ :



R_{can} :

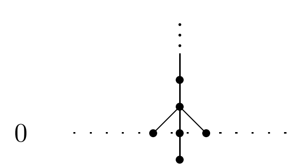


Here the two minimal points of R_{can} correspond to the zero cycle and to Artin's fundamental cycle.

3.4.4. Classification. Continuation of the Artin–Laufer program. By the 'Artin-Laufer program' we understand results in the spirit of subsections 2.5 and 2.6 (i.e., results which provide, say, the Hilbert–Samuel function and the geometric genus for a topologically identified family). Although we know some obstructions to continue this type of results (see e.g. [33]), we still hope in its future.

In order to continue this program, we first have to identify subfamilies for which one can show that they share 'common properties'. We propose to identify these subfamilies by graded roots: For each fixed possible graded root, we consider all the singularities which have their canonical graded root identical with the fixed one. We believe that this invariant is exactly the right object which (conjecturally) guides the topological classification of singularities. (As for starting point, see 3.4.1 and 3.4.2.)

Let us formulate here a very precise, new situation/family. Let us say that a singularity is of 'general type' if it is not rational or elliptic. This happens if and only if $\chi(Z_{min}) < 0$. Now, the subfamily which collects the *'simplest' general type singularities* is characterized by the fact that they share the same canonical graded root. This is the following:



It is not difficult to see that for this class $2 \leq p_g \leq 3$, and $p_g = 3$ if and only if the singularity is Gorenstein. We expect that all the results of Laufer for minimally elliptic singularities (or, those of [32]) will have their analogs in *minimally general* case as well.

Notice that the class of 'minimally general' singularities is not empty, e.g. it contains $x^5 + y^5 + z^3$. But in general, it is not clear at all what abstract graded roots can be realized as canonical graded roots associated with singularities (even when we consider only numerical Gorenstein singularities, when the canonical root has a $\mathbb{Z}_2$ symmetry).

3.4.5. Problem. *Determine all the possible canonical graded roots.*

Notice that the possible resolution graphs are characterized by Grauert criterion, namely that they are negative definite. For each negative definite graph (tree) we construct the graded root. The problem is to find a combinatorial characterization of all of them.

3.5 Graded roots and Heegaard–Floer homology of almost rational graphs.

3.5.1. This subsection contains two important results. The first one says that for a plumbed 3–manifolds obtained from an AR plumbing graph, the associated Heegaard–Floer homologies can be determined purely combinatorially from the plumbing graph via the homology of the corresponding graded roots. More precisely, theorems (4.8) and (8.3) of [34] read as follows (we invite the reader to recall the definition of the distinguished representatives k_r in 2.2.7):

3.5.2. Theorem. *Assume that* Γ *is an AR-graph. Then, for any* $[k]\in Spin^c(M)$

$$HF^+_{odd}(-M,[k])=0,$$

and

$$HF^+_{even}(-M,[k])=\mathbb{H}(R_{k_r},\chi_{k_r})\Big[-\frac{k_r^2+\#\mathcal{J}}{4}\Big].$$

In particular,

$$d(-M,[k])=-\max_{k'\in[k]}\frac{(k')^2+\#\mathcal{J}}{4}=-\frac{k_r^2+\#\mathcal{J}}{4}+2\min\chi_{k_r}.$$

This is generalization of the main result of [51] (where the statement was proved for 'almost minimal rational' graphs).

3.5.3. Corollary. *If* Γ *is an AR-graph, and* $[k]\in Spin^c(M)$*, then:*

$$\mathbf{sw}^{OSZ}(-M,[k])=\frac{k_r^2+\#\mathcal{J}}{8}+\mathrm{rank}_{\mathbb{Z}}\,\mathbb{H}_{red}(R_{k_r},\chi_{k_r})-\min\chi_{k_r}.$$

3.5.4. We wish to emphasize that in most of the applications, the root (R_{k_r},χ_{k_r}) is *not* determined by its definition 3.3.1, but by an algorithm which will be described in the sequel. This algorithm, the second main result of the subsection, is motivated by singularity theory; its idea was suggested by different computational sequences (as generalizations of Laufer's computation sequence whose output is the Artin's minimal cycle Z_{min}), used by Laufer, Stephen Yau and others in the combinatorics of elliptic (or other) singularities.

In order to present the algorithm, we distinguish one of vertices of our AR-graph Γ, which satisfies the definition of AR-graphs. By convention, it corresponds to the index $0\in\mathcal{J}$ (fixed for ever), cf. 2.7.1.

For a rational cycle $x=\sum_j r_jE_j\in L'$ we write $pr_j(x)$ for the coefficient r_j.

3.5.5. The definition of the sequence $\{x_{[k]}(i)\}_{i\geq 0}$**.** We fix a class $[k]$. Recall the definition of $l'_{[k]}$ from 2.2.7. Then for any integer $i\geq 0$ there exists a unique cycle $x_{[k]}(i)\in L$ with the following properties:

(a) $pr_0(x_{[k]}(i))=i$;

(b) $(x_{[k]}(i)+l'_{[k]},E_j)\leq 0$ for any $j\in\mathcal{J}\setminus\{0\}$;

(c) $x_{[k]}(i)$ is minimal (with respect the partial ordering $\leq$) with the properties (a) and (b).

(Part (b) for the canonical class $[k]=[K]$ reads as $(x_{can}(i),E_j)\leq 0$ for any $j\neq 0$; notice the similarity with the definition of Artin's minimal cycle.)

3.5.6. Theorem. ((9.3) of [34]) *Fix a spin^c-structure* $[k]$*. There exists an integer* T_0 *(which depend on* $[k]$*) such that* $\chi_{k_r}(x_{[k]}(i+1))\geq\chi_{k_r}(x_{[k]}(i))$ *for any* $i\geq T_0$*. Moreover, the graded root associated with* $\tau_{[k]}:\{0,\dots,T_0\}\to\mathbb{N}$ *given by* $\tau_{[k]}(i):=\chi_{k_r}(x_{[k]}(i))$ *satisfies*

$$(R_{k_r},\chi_{k_r})=(R_{\tau_{[k]}},\chi_{\tau_{[k]}}).$$

3.5.7. Remarks. (a) In general, it is not easy to find the cycles $x_{[k]}(i)$. Fortunately, one does not need all the coefficients of these cycles, only the values $\tau_{[k]}(i) = \chi_{k_r}(x_{[k]}(i))$. In most of the cases they are computed inductively using the following equality (cf. (9.1)(c) [34]):

$$\chi_{k_r}(x_{[k]}(i+1)) = \chi_{k_r}(x_{[k]}(i) + E_0).$$

Since the right hand side is $\chi_{k_r}(x_{[k]}(i)) + 1 - (E_0, l'_{[k]} + x_{[k]}(i))$, basically one needs only the intersections $(E_0, x_{[k]}(i))$ for any i.

(b) Clearly $\tau_{[k]}(0) = 0$. One also shows that $\tau_{[k]}(1) > 0$, hence 3.2.6 can be applied:

$$\begin{aligned} \mathrm{rank}_{\mathbb{Z}}\, \mathbb{H}_{red}(R_{k_r}, \chi_{k_r}) - \min \chi_{k_r} &= \textstyle\sum_{i\geq 0} \max\big(0\,,\, \tau_{[k]}(i) - \tau_{[k]}(i+1)\big) \\ &= \textstyle\sum_{i\geq 0} \max\big(0\,,\, -1 + (E_0, l'_{[k]} + x_{[k]}(i))\big). \end{aligned}$$

3.6 Example. Lens spaces.

3.6.1. Notations. In this subsection M is the lens space $L(p,q)$, for its plumbing graph see 2.4.3, whose notations we will preserve. M can also be obtained as a $-p/q$ surgery on the unknot in S^3.

We invite the reader to refresh the notations of 2.2.1; in particular, we recall that $\{D_j\}_{j\in\mathcal{J}}$ denotes the dual base in L'. For any $1 \leq i \leq s$ we write the continued fraction $[k_i, \ldots, k_s]$ as a rational number n_{is}/d_{is} with $n_{is} > 0$ and $gcd(n_{is}, d_{is}) = 1$. E.g., $n_{1s} = p$ and $n_{2s} = q$. Consider also q' defined by $qq' \equiv 1$ modulo p and $0 < q' < p$. ($\{x\} := x - [x]$ denotes the fractional part of the real number x.)

3.6.2. The group H, the *spin*^{c} structures, and the elements $l'_{[k]}$. We write $[D_j]$ for the class $D_j + L$ of D_j in $L'/L = H$. In the present case, $H = \mathbb{Z}_p$, and $[D_s]$ is one of its generators. In fact, $[D_j] = [n_{j+1,s}D_s]$ in H $(1 \leq j \leq s)$.

Similarly, the set of *spin*^{c}-structures on M is the set of orbits $\{[-aD_s]\}_{0\leq a<p} = \{-aD_s + L\}_{0\leq a<p}$ (we prefer to use this sign, since $-D_s$ is effective). More precisely, this correspondence is $[k] = K + 2(-aD_s + L)$, where a runs from 0 to $p-1$. In order to emphasize the role of a, we also use the notation $l'_{[-aD_s]}$ for $l'_{[k]}$. For any $0 \leq a < p$ write

$$l'_{[-aD_s]} = -(a_1D_1 + a_2D_2 + \cdots + a_sD_s).$$

The next discussion clarifies the relation between the integer $0 \leq a < p$ (which codifies $Spin^c(L(p,q))$) and the system $S(a) := (a_1, \ldots, a_s)$ (the coefficients of the corresponding minimal vectors $l'_{[-aD_s]}$).

Using the definition of $l'_{[-aD_s]}$ and the combinatorics of the continued fractions, one shows (for details, see [34], § 10) that the entries of $(a_1, \ldots, a_s)$ satisfy the system of inequalities:

$$(SI) \qquad \begin{cases} a_1 \geq 0, \ldots, a_s \geq 0 \\ n_{i+1,s}a_i + n_{i+2,s}a_{i+1} + \cdots + n_{ss}a_{s-1} + a_s < n_{is} \quad \text{for any } 1 \leq i \leq s. \end{cases}$$

By this system one can identify the integers $0 \leq a < p$ with the possible combinations $(a_1, \ldots, a_s)$ satisfying (SI). Indeed, the integer a can be recovered from $(a_1, \ldots, a_s)$ by

$$a = n_{2s}a_1 + n_{3s}a_2 + \cdots + n_{ss}a_{s-1} + a_s. \tag{1}$$

And, any $0 \leq a < p$ determines inductively the entries $a_1, \cdots, a_s$ by the formula

$$a_i = \Big[\frac{a - \sum_{t=1}^{i-1} n_{t+1,s}a_t}{n_{i+1,s}}\Big] \quad (1 \leq i \leq s).$$

3.6.3. As a curiosity, we mention that the above system (SI) can also be interpreted in language of 'generalized' continued fractions. For any $1 \leq i \leq s$ write $r_i := n_{is}/n_{i+1,s}$. Then

$$\frac{a}{p} = \cfrac{a_1 + \cfrac{a_2 + \cfrac{\cdots + \cfrac{a_{s-1} + \cfrac{a_s}{r_s}}{\cdots}}{r_3}}{r_2}}{r_1}.$$

The inequalities (SI) imply that all the possible fractions in the above expression are < 1; and this property guarantees the uniqness of the entries $(a_1, \dots, a_s)$ in this continued fraction (for any given $0 \leq a < p$).

3.6.4. In concrete examples, the fastest way to determine $S(a) = (a_1, \dots, a_s)$, $0 \leq a < p$, in the order $S(p-1), \dots, S(0)$, is the following. Start with $S(p-1) = (k_1 - 1, k_2 - 2, \dots, k_s - 2)$. Assume that $S(a) = (a_1, \dots, a_s)$ is already determined. Then, if $a_s > 0$, then $S(a-1) = (a_1, \dots, a_{s-1}, a_s - 1)$. If $a_i = \cdots = a_s = 0$, but $a_{i-1} \neq 0$, then

$$S(a-1) = (a_1, \dots, a_{i-2}, a_{i-1} - 1, k_i - 1, k_{i+1} - 2, \dots, k_s - 2).$$

3.6.5. The Ozsváth-Szabó Heegaard–Floer homology $HF^+(\pm M)$. Since Γ is rational, by 3.4.1 and 3.5.2 (see also 2.3.2), $HF^+_{red}(\pm M) = 0$, and $HF^+(\pm M, [k]) = \mathcal{T}^+_{\pm d}$, and $\mathbf{sw}^{OSZ}(M, [k]) = -d/2$, where

$$d := d(M, [k]) = \frac{k_r^2 + s}{4} = \frac{K^2 + s}{4} - 2\chi(l'_{[k]}).$$

Notice that (see e.g. (7.1) of [37]):

$$\frac{K^2 + s}{4} = \frac{p-1}{2p} - 3 \cdot \boldsymbol{s}(q, p),$$

where $\boldsymbol{s}(q, p)$ denotes the Dedekind sum

$$\boldsymbol{s}(q, p) = \sum_{l=0}^{p-1} \Big(\Big(\frac{l}{p}\Big)\Big)\Big(\Big(\frac{ql}{p}\Big)\Big), \text{ where } ((x)) = \begin{cases} \{x\} - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

(In fact, this formula for $K^2 + s$ for cyclic quotients goes back to the work of Hirzebruch.)

Finally, we determine $\chi(l'_{[-aD_s]})$.

3.6.6. Proposition. *For any* $0 \leq a < p$ *one has:*

$$\chi(l'_{[-aD_s]}) = \frac{a(1-p)}{2p} + \sum_{j=1}^{a} \Big\{\frac{jq'}{p}\Big\}.$$

3.6.7. Clearly, with the choice D_1 as a generator, one has

$$\chi(l'_{[-aD_1]}) = \frac{a(1-p)}{2p} + \sum_{j=1}^{a} \Big\{\frac{jq}{p}\Big\}.$$

3.6.8. Remark. The Casson-Walker invariant is $\lambda(L(p, q)) = p \cdot \boldsymbol{s}(q, p)/2$ (see e.g. [37](5.3)). It satisfies

$$-\lambda(M) = \sum_{[k] \in Spin^c(M)} \frac{k_r^2 + s}{8},$$

or, equivalently:

$$\sum_{[k]} \chi(l'_{[k]}) = \frac{p-1}{4} - p \cdot \boldsymbol{s}(q, p).$$

4 Line bundles associated with surface singularities [35].

4.1 Introduction

In [37] L. Nicolaescu and the author formulated a conjecture which relates the geometric genus of a complex analytic normal surface singularity (X,0) (whose link M is a rational homology sphere) with the Seiberg-Witten invariant of M associated with the 'canonical' $spin^c$ structure of M. This is a generalization of a conjecture of Neumann and Wahl [43] formulated for complete intersection singularities with integral homology sphere links. The interested reader can find in the articles [37, 38, 39, 34, 33] the verification of the conjecture in different cases; in [26] some counterexamples, showing that the original assumptions of [37] (namely, that $(X,0)$ is $\mathbb{Q}$–Gorenstein with $\mathbb{Q}HS$ *link) are too weak (hence one needs to consider some additional restrictions in order to give a chance to the conjecture). For related constructions and results, see also the articles of Neumann and Wahl [44, 45, 46].*

Since the Seiberg-Witten theory of the link M provides a rational number for any $spin^c$ *structure (which are classified by $H_1(M,\mathbb{Z})$), it was a natural challenge to search for a complete set of conjecturally valid identities, which involve all the Seiberg-Witten invariants (giving an analytic – i.e. singularity theoretical – interpretation of them).*

The formulation of this set of identities/properties — conjecturally valid for some 'nice' families of normal surface singularities — is one of the goals of the present section. In fact (similarly as in [37]), we formulate conjecturally valid inequalities which hopefully become equalities in special rigid situations. In this way, the Seiberg-Witten invariants provide optimal topological upper bounds for the dimensions of the first sheaf-cohomology of some line bundles living on a resolution of $(X,0)$.

The first part of the section is devoted to the construction and study of these 'natural' holomorphic line bundles on the resolution $\tilde{X}$. This construction automatically provides a natural splitting of the exact sequence

$$0 \to Pic^0(\tilde{X}) \to Pic(\tilde{X}) \xrightarrow{c_1} H^2(\tilde{X},\mathbb{Z}) \to 0.$$

The line-bundle construction is compatible with abelian covers. This allows us to reformulate the 'conjecture' in its second version, which relates the echivariant geometric genus of the universal (unbranched) abelian cover of $(X,0)$ with the Seiberg-Witten invariants of the link.

In the last subsection we verify the validity of the conjectured properties for rational singularities.

The presentation is based on the author's unpublished preprint [35]. Some h^1–computations for the case of rational singularities were also found independently by T. Okuma [48].

4.2 Line bundles on $\tilde{X}$.

4.2.1. Let $\pi : (\tilde{X}, E) \to (X,0)$ be a fixed good resolution of $(X,0)$. Similarly as above, we assume that the link M is a $\mathbb{Q}HS$, i.e. $H^1(\tilde{X},\mathbb{Z}) = 0$. Therefore, one has the exact sequence

$$0 \to Pic^0(\tilde{X}) \to Pic(\tilde{X}) \xrightarrow{c_1} L' \to 0, \tag{1}$$

where $Pic^0(\tilde{X}) = H^1(\tilde{X},\mathcal{O}_{\tilde{X}})$, $Pic(\tilde{X}) = H^1(\tilde{X},\mathcal{O}^*_{\tilde{X}})$ (= isomorphism classes of holomorphic line bundles on $\tilde{X}$), and $c_1(\mathcal{L}) = \sum_j \deg(\mathcal{L}|E_j)\, D_j$ is the set of Chern numbers (multidegree) of $\mathcal{L}$.

In the sequel, we will use the same notation for $l = \sum n_j E_j \in L$ and the algebraic cycle $\sum n_j E_j$ of $\tilde{X}$ supported by E. E.g., for any $l = \sum n_j E_j \in L$ one can take the line bundle $\mathcal{O}_{\tilde{X}}(l) := \mathcal{O}_{\tilde{X}}(\sum n_j E_j)$. Notice that $c_1(\mathcal{O}_{\tilde{X}}(l)) = l$, hence c_1 admits a group-section $s_L : L \to Pic(\tilde{X})$ above the subgroup L of L'. The main goal of this subsection is to construct a (natural) group section $s : L' \to Pic(\tilde{X})$ of (1) which extends s_L (and is compatible with abelian coverings).

Clearly, if $Pic^0(\tilde{X}) = 0$ (i.e. if $(X,0)$ is rational, see [3], or §2.5 here) then there is nothing to construct: c_1 is an isomorphism, and $s := c_1^{-1}$ identifies the lines bundles with their multidegree. Nevertheless, for non-rational singularities, even the existence of any kind of splitting of (1) is not so obvious.

4.2.2. The first construction. Notice that $\tilde{X} \setminus E \approx X \setminus \{0\}$ has the homotopy type of M, hence the abelianization map $\pi_1(\tilde{X} \setminus E) = \pi_1(M) \to H$ defines a regular Galois covering of $\tilde{X} \setminus E$. This has a unique extension $c : Z \to \tilde{X}$ with Z normal and c finite [13]. The (reduced) branch locus of c is included in E, and the Galois action of H extends to Z as well. Since E is a normal crossing divisor, the only singularities what Z might have are cyclic quotient singularities (situated above $Sing(E)$). Let $r : \tilde{Z} \to Z$ be a resolution of these singular points of Z, such that $(c \circ r)^{-1}(E)$ is a normal crossing divisor.

Let $(X_a, 0)$ denote the the universal (unbranched) abelian cover of $(X, 0)$, i.e. the unique normal singular germ corresponding to the regular covering of $X \setminus \{0\}$ associated with $\pi_1(X \setminus \{0\}) \to H$. Notice that $\tilde{Z}$ is a good resolution of $(X_a, 0)$. Hence, one can consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & L & \to & L' & \to & H & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p_H} & & \\ 0 & \to & L_a & \to & L'_a & \to & H_a & \to & 0 \end{array}$$

Here, the first, resp. second, horizontal line is the exact sequence 2.2.1(3) applied for the resolution $\tilde{X} \to X$, resp. for $\tilde{Z} \to X_a$. The vertical arrows (pull-back of cohomology classes) are induced by $p = c \circ r$.

4.2.3. Lemma. $p_H = 0$. *In particular, $p'(L') \subset L_a$ (i.e. any element $p'(l')$, $l' \in L'$, can be represented by a divisor supported by the exceptional divisor in $\tilde{Z}$).*

Proof. Denote by M_a the link of $(X_a, 0)$. The morphism $p_H : H^2(M, \mathbb{Z}) \to H^2(M_a, \mathbb{Z})$ is dual to $p_* : H_1(M_a, \mathbb{Z}) \to H_1(M, \mathbb{Z})$, which is zero since $H_1(M_a, \mathbb{Z})$ is the abelianization of the commutator subgroup of $\pi_1(M)$. □

This shows that for any $l' \in L'$, one can take $\mathcal{O}_{\tilde{Z}}(p'(l')) \in Pic(\tilde{Z})$.

4.2.4. Theorem. *The line bundle $\mathcal{O}_{\tilde{Z}}(p'(l'))$ is a pull-back of a unique element of $Pic(\tilde{X})$.*

Proof. We break the proof into several steps. Let $f : S \to T$ be one of the maps $r : \tilde{Z} \to Z$, $c : Z \to \tilde{X}$ or $p : \tilde{Z} \to \tilde{X}$. In each case one has a commutative diagram of type

$$\begin{array}{ccccccccc} H^1(T, \mathbb{Z}) & \to & Pic^0(T) & \to & Pic(T) & \to & H^2(T, \mathbb{Z}) & \to & 0 \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f^{0,*}} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f'} & & \\ H^1(S, \mathbb{Z}) & \to & Pic^0(S) & \to & Pic(S) & \to & H^2(S, \mathbb{Z}) & \to & 0 \end{array}$$

(I). *Assume that $f = c$.* Then:

- *c' is injective* since $c' \otimes \mathbb{Q} : H^2(\tilde{X}, \mathbb{Q}) \approx H^2(Z, \mathbb{Q})^H \hookrightarrow H^2(Z, \mathbb{Q})$ is so and $H^2(\tilde{X}, \mathbb{Z})$ is free.
- *$c^{0,*}$ is injective with image $Pic^0(Z)^H$.* Indeed (see also the proof of 4.2.9), $c_*\mathcal{O}_Z$ has a direct sum decomposition $\oplus_\chi \mathcal{L}_\chi$ into line bundles $\mathcal{L}_\chi$, where the sum runs over all the characters of H, and for the trivial character $\mathcal{L}_1 = \mathcal{O}_{\tilde{X}}$. Therefore, since c is finite, $H^1(Z, \mathcal{O}_Z) = H^1(\tilde{X}, c_*\mathcal{O}_Z) = \oplus_\chi H^1(\tilde{X}, \mathcal{L}_\chi)$, whose H-invariant part is $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$.
- $\mathrm{Im}(H^1(Z, \mathbb{Z}) \to Pic^0(Z)) \cap \mathrm{Im}p^{0,*} = 0$. Indeed, since any element of $\mathrm{Im}p^{0,*}$ is H-invariant, the above intersection is in $\mathrm{Im}(H^1(Z, \mathbb{Z})^H \to Pic^0(Z)^H)$. But $H^1(Z, \mathbb{Z})^H$ embeds into $H^1(Z, \mathbb{Q})^H = H^1(\tilde{X}, \mathbb{Q}) = 0$, hence it is trivial.
- *c^* is injective,* a fact which follows from the previous three statements.

(II). *Assume that $f = r$.* Then r'' is an isomorphism (since such a resolution does not modify $H_1(\cdot, \mathbb{Z})$ of the exceptional divisors), $r^{0,*}$ is an isomorphism (since a quotient singularity has geometric genus zero), and r' is injective. Hence (by a diagram check) r^* is also injective.

(III). *Assume that $f = p = c \circ r$.* Then:

- *For any $l' \in L'$, $\mathcal{O}_{\tilde{Z}}(p'(l'))$ is in the image of p^*.* Indeed, take a line bundle $\mathcal{L} \in Pic(\tilde{X})$ with $c_1\mathcal{L} = l'$. Then $\mathcal{L}' := \mathcal{O}_{\tilde{Z}}(p'(l')) \otimes p^*\mathcal{L}^{-1}$ has trivial multidegree, and it is in $Pic^0(\tilde{Z})^H$. But using (I-II) $p^{0,*}$ is onto on $Pic^0(\tilde{Z})^H$, hence $\mathcal{L}'$ is a pull-back. Hence $\mathcal{O}_{\tilde{Z}}(p'(l'))$ itself is a pull-back.
- Again, from (I-II), p^* *is injective*; a fact which ends the proof of 4.2.4. □

4.2.5. Notation. Write $\mathcal{O}_{\tilde{X}}(l')$ for the unique line bundle $\mathcal{L} \in Pic(\tilde{X})$ with $p^*(\mathcal{L}) = \mathcal{O}_{\tilde{Z}}(p'(l'))$.

The proof of 4.2.4 also implies the following fact.

4.2.6. Corollary. *$s : L' \to Pic(\tilde{X})$ defined by $l' \mapsto \mathcal{O}_{\tilde{X}}(l')$ is a group section of the exact sequence 4.2.1(1) which extends s_L (cf. 4.2.1).*

4.2.7. Example. Consider an A_2-singularity. Then the exceptional divisor E of the minimal resolution $\tilde{X}$ contains two components E_1 and E_2 which intersect each other transversely, both with self-intersection -2. $H=\mathbb{Z}_3$, and Z also contains two divisors, but they intersect each other in a singular point whose resolution consists of a -3 curve. Hence $\tilde{Z}$ contains a string of three rational curves F_1, F_{12} and F_2 with self-intersections -1, -3 and -1. Then $p'(E_1)=3F_1+F_{12}$ and $p'(E_2)=F_{12}+3F_2$. Since $-D_1=(2E_1+E_2)/3$, one gets that $p'(-D_1)=2F_1+F_{12}+F_2$, hence it is an integral cycle. In particular, $c_1^{-1}(-D_1)$ is that line bundle whose p^*-image is $\mathcal{O}_{\tilde{Z}}(2F_1+F_{12}+F_2)$.

4.2.8. The second construction. The following result will illuminate a different aspect of the line bundles $\mathcal{O}_{\tilde{X}}(l')$. In fact, the next proposition could also serve as the starting point of a different construction of these line bundles.

Below we will write $\hat{H}$ for the Pontrjagin dual $\mathrm{Hom}(H,S^1)$ of H. Recall that the natural map

$$\theta: H\to \hat{H}, \quad \text{induced by} \quad [l']\mapsto e^{2\pi i(l',\cdot)}$$

is an isomorphism.

4.2.9. Theorem. *Consider the finite covering $c: Z\to\tilde{X}$, and set $Q:=\{\sum r_jE_j\in L' : 0\leq r_j<1\}\subset L'$ (cf. 2.2.7) as above. Then the H-eigenspace decomposition of $c_*\mathcal{O}_Z$ has the form:*

$$c_*\mathcal{O}_Z=\oplus_{\chi\in\hat{H}}\mathcal{L}_\chi,$$

where $\mathcal{L}_{\theta(h)}=\mathcal{O}_{\tilde{X}}(-l'_e(h))$ for any $h\in H$. In particular, $c_*\mathcal{O}_Z=\oplus_{l'\in Q}\mathcal{O}_{\tilde{X}}(-l')$.

Proof. The proof is based on a similar statement of Kollár valid for cyclic coverings, see e.g. [18], §9.

First notice that $c_*\mathcal{L}_Z$ is free. Indeed, since all the singularities of Z are cyclic quotient singularities, this fact follows from the corresponding statement for cyclic Galois coverings, which was verified in [18]. Moreover, above $\tilde{X}\setminus E$ the covering is regular (unbranched) corresponding to the regular representation of H. Therefore, $\mathrm{rank}\, c_*\mathcal{L}_Z=|H|$, and it has an eigenspace decomposition $\oplus_\chi\mathcal{L}_\chi$, where all the characters $\chi\in\hat{H}$ appear, and $\mathcal{L}_\chi|\tilde{X}\setminus E$ is a line bundle. By a similar reduction as above to the cyclic case, one gets that $\mathcal{L}_\chi$ itself is a line bundle. Moreover, $\mathcal{L}_1$ (corresponding to the trivial character 1) equals $\mathcal{O}_{\tilde{X}}$.

Next we identify $\mathcal{L}_\chi$ for any character. Fix $\chi_0\in\hat{H}$. It generates the subgroup $\langle\chi_0\rangle$ in $\hat{H}$. Write $\hat{K}:=\hat{H}/\langle\chi_0\rangle$. The morphism $\pi_1(M)\to H\to\langle\chi_0\rangle\hat{}$ defines a well-defined normal space Y as a cyclic Galois $\langle\chi_0\rangle\hat{}$-covering of $\tilde{X}$. Clearly, one has the natural maps $Z\xrightarrow{f}Y\xrightarrow{e}\tilde{X}$ with $e\circ f=c$. By similar argument as above $f_*\mathcal{O}_Z=\oplus_{\xi\in\hat{K}}\bar{\mathcal{L}}_\xi$ for some $\bar{\mathcal{L}}_\xi\in Pic(Y)$ and $\bar{\mathcal{L}}_1=\mathcal{O}_Y$. Since $e_*\bar{\mathcal{L}}_\xi=\oplus_{[\chi]=\xi}\mathcal{L}_\chi$, and $[\chi_0]=0$ in $\hat{K}$, one gets that $\mathcal{L}_{\chi_0}$ is one of the summands of $e_*\bar{\mathcal{L}}_1=e_*\mathcal{O}_Y$. In particular, $\mathcal{L}_{\chi_0}$ can be recovered from the cyclic covering e as the χ_0-eigenspace of $e_*\mathcal{O}_Y$.

Assume that the order of χ_0 is n, i.e. $\langle\chi_0\rangle=\mathbb{Z}_n$, and we regard χ_0 as the distinguished generator of $\mathbb{Z}_n$. Write $g_j:=[\partial D_j]\in H$. Then for any $j\in\mathcal{J}$, $\chi_0(g_j)$ has the form $e^{2\pi i m_j/n}$ for some (unique) $0\leq m_j<n$. Using these integers, define the divisor $B:=\sum_j m_jE_j$. Since for any fixed $i\in\mathcal{J}$ one has

$$1=\chi_0([E_i])=\chi_0(\textstyle\sum_j I_{ij}g_j)=e^{2\pi i(\sum_j m_jE_j,E_i)/n},$$

one gets that $B/n\in L'$. Let $h=[B/n]$ be its class in H. Clearly, $\theta(h)=\chi_0$ since

$$\theta(h)(g_j)=e^{2\pi i(\sum m_jE_j/n,D_j)}=e^{2\pi i m_j/n}=\chi_0(g_j).$$

On the other hand, by [18], 9.8 (and from the fact that all the coefficients of B/n are in the interval $[0,1)$), the χ_0-eigenspace of $e_*\mathcal{O}_Y$ is some line bundle $\mathcal{L}^{-1}$ with the properties (i) $\mathcal{L}^{\otimes n}=\mathcal{O}_{\tilde{X}}(B)$ and (ii) $e^*\mathcal{L}=\mathcal{O}_Y(e'(B/n))$. From (i) follows that $c_1(\mathcal{L})=B/n$, hence (ii), via our definition 4.2.5, reads as $\mathcal{L}=\mathcal{O}_{\tilde{X}}(B/n)$. Therefore, $\mathcal{L}^{-1}=\mathcal{O}_{\tilde{X}}(-B/n)$. Since B/n is in the unit cube, $B/n=l'_e(h)$. $\square$

4.3 Some cohomological computations

4.3.1. Let $(X,0)$ and $\pi:\tilde X\to X$ be as above. In this section we analyse $h^1(\tilde X,\mathcal{L})$ for any $\mathcal{L}\in Pic(\tilde X)$.

First, recall the following general (Kodaira, or Grauert-Riemenschneider type) vanishing theorem (cf. [55], page 119, Ex. 15):

If $c_1(\mathcal{L})\in K+L_{\mathbb{Q},ne}$, then $h^1(l,\mathcal{L}|_l)=0$ for any $l\in L$, $l>0$, hence $h^1(\tilde X,\mathcal{L})=0$.

The next statement is an improvement of it, valid for rational singularities:

4.3.2. *Assume that (X,0) is a rational singularity. If $c_1(\mathcal{L})\in L_{\mathbb{Q},ne}$, then $h^1(l,\mathcal{L}|_l)=0$ for any $l>0$, $l\in L$, hence $h^1(\tilde X,\mathcal{L})=0$ too.*

Proof. For any $l>0$ there exists $E_j\subset|l|$ such that $(E_j,l+K)<0$. Indeed, $(E_j,l+K)\geq 0$ for any j would imply $\chi(l)=-(l,l+K)/2\leq 0$, which would contradict the rationality of (X,0) [3]. Then, from the cohomology exact sequence of

$$0\to\mathcal{L}\otimes\mathcal{O}_{E_j}(-l+E_j)\to\mathcal{L}|_l\to\mathcal{L}|_{l-E_j}\to 0$$

one gets $h^1(\mathcal{L}|_l)=h^1(\mathcal{L}|_{l-E_j})$, hence by induction $h^1(\mathcal{L}|_l)=0$. □

We will generalize this proof as follows.

4.3.3. Proposition. *Let $\tilde X\to X$ be a good resolution of a normal singularity (X,0) as above.*

(a) For any $l'\in L'$ there exists a unique minimal element $l_{l'}\in L_e$ with $e(l'):=l'-l_{l'}\in L_{\mathbb{Q},ne}$.

(b) $l_{l'}$ can be found by the following (generalized Laufer's) algorithm. One constructs a "computation sequence" $x_0,x_1,\ldots,x_t\in L_e$ with $x_0=0$ and $x_{i+1}=x_i+E_{j(i)}$, where the index $j(i)$ is determined by the following principle. Assume that x_i is already constructed. Then, if $l'-x_i\in L_{\mathbb{Q},ne}$, then one stops, and $t=i$. Otherwise, there exists at least one j with $(l'-x_i,E_j)<0$. Take for $j(i)$ one of these j's. Then this algorithm stops after a finitely many steps, and $x_t=l_{l'}$.

(c) For any $\mathcal{L}\in Pic(\tilde X)$ with $c_1(\mathcal{L})=l'$ one has:

$$h^1(\mathcal{L})=h^1(\mathcal{L}\otimes\mathcal{O}_{\tilde X}(-l_{l'}))-(l',l_{l'})-\chi(l_{l'}).$$

In particular (since $c_1(\mathcal{L}\otimes\mathcal{O}_{\tilde X}(-l_{l'}))\in L_{\mathbb{Q},ne}$), the computation of any $h^1(\mathcal{L})$ can be reduced (modulo the combinatorics of $(L,(\cdot,\cdot))$) to the computation of some $h^1(\mathcal{L}')$ with $c_1(\mathcal{L}')\in L_{\mathbb{Q},ne}$.

Proof. (a) First notice that since B is negative definite, there exists at least one effective cycle l with $l'-l\in L_{\mathbb{Q},ne}$ (take e.g. a large multiple of some x with $(x,E_j)<0$ for any j). Next, we prove that if $l'-l_i\in L_{\mathbb{Q},ne}$ for $l_i\in L_e$, $i=1,2$, and $l:=\min\{l_1,l_2\}$, then $l'-l\in L_{\mathbb{Q},ne}$ as well. For this, write $x_i:=l_i-l\in L_e$. Then $|x_1|\cap|x_2|=\emptyset$, hence for any fixed j, $E_j\not\subset|x_i|$ for at least one of the i's. Therefore, $(l'-l,E_j)=(l'-l_i,E_j)+(x_i,E_j)\geq 0$.

(b) First we prove that $x_i\leq l_{l'}$ for any i. For $i=0$ this is clear. Assume that it is true for some i but not for $i+1$, i.e. $E_{j(i)}\not\subset|l_{l'}-x_i|$. But this would imply $(l'-x_i,E_{j(i)})=(l'-l_{l'},E_{j(i)})+(l_{l'}-x_i,E_{j(i)})\geq 0$, a contradiction. The fact that $x_i\leq l_{l'}$ for any i implies that the algorithm must stop, and $x_t\leq l_{l'}$. But then by the minimality of $l_{l'}$ (part a) $x_t=l_{l'}$. (cf. [20].)

(c) For any $0\leq i<t$, consider the exact sequence

$$0\to\mathcal{L}\otimes\mathcal{O}_{\tilde X}(-x_{i+1})\to\mathcal{L}\otimes\mathcal{O}_{\tilde X}(-x_i)\to\mathcal{L}\otimes\mathcal{O}_{E_{j(i)}}(-x_i)\to 0.$$

Since $\deg(\mathcal{L}\otimes\mathcal{O}_{E_{j(i)}}(-x_i))=(l'-x_i,E_{j(i)})<0$, one gets $h^0(\mathcal{L}\otimes\mathcal{O}_{E_{j(i)}}(-x_i))=0$. Therefore

$$h^1(\mathcal{L}\otimes\mathcal{O}_{\tilde X}(-x_i))-h^1(\mathcal{L}\otimes\mathcal{O}_{\tilde X}(-x_{i+1}))=-\chi(\mathcal{L}\otimes\mathcal{O}_{E_{j(i)}}(-x_i))$$

which equals $-(l',x_{i+1}-x_i)+\chi(x_i)-\chi(x_{i+1})$. Hence the result follows by induction. □

4.3.4. Examples. If $\mathcal{L}=\mathcal{O}_{\tilde{X}}(l')$ for some $l'\in L'$ (cf. 4.2.5) then 4.3.3(c) reads as

$$h^1(\mathcal{O}_{\tilde{X}}(l')))=h^1(\mathcal{O}_{\tilde{X}}(e(l')))-(l',l_{l'})-\chi(l_{l'}).$$

Additionally, if $(X,0)$ is rational then $h^1(\mathcal{O}_{\tilde{X}}(e(l')))=0$ by 4.3.2, hence $h^1(\mathcal{O}_{\tilde{X}}(l'))=-(l',l_{l'})-\chi(l_{l'})$. In particular, *for (X,0) rational, $h^1(\mathcal{L})$ depends only on topological data and it is independent of the analytic structure of (X,0)* .

4.3.5. Definition. We will distinguish the following set of rational cycles:

$$\mathbb{L}':=\{l'\in L'\,:\,e(l')=l'_{ne}(h)\text{ for some }h\in H\}=\bigcup_{h\in H}l'_{ne}(h)+L_e.$$

It is easy to verify the following inclusions: $L'_e\subset\bigcup_{l'\in Q}-l'+L_e\subset\mathbb{L}'$.

4.4 Main (conjectured) properties.

4.4.1. Preliminary words. The next 'conjecture'/expected property is a generalization of the conjecture of [37], where only the case of canonical $spin^c$ structure was considered. The conjectured property provides an optimal upper bound for $h^1(\mathcal{L})$ $(\mathcal{L}\in Pic(\tilde{X}))$ in terms of the topology of $(X,0)$ and $c_1(\mathcal{L})$. The topological ingredient is provided by the Seiberg-Witten theory of the link.

Now, the author knows that the conjecture of [37] is not true in the high generality in which it was formulated, see [26] and [33] for details. Nevertheless, we expect that it is true for a large class of normal surface singularities (subclass of $\mathbb{Q}$-Gorenstein singularities with $\mathbb{Q}HS$ *links). In this article we will not enter in the guess of the (largest) possible class for which the property is valid; we just formulate these conjectures as expected properties. But the reader might consider them as conjectures valid for, say (jokingly), NYI ('not-yet-identified') singularities.*

We will present two versions ('strong' and 'weak'), both of them having two parts. The first ones, (SIn) *and* (WIn)*, are* inequalities*: we expect their validity for a class of singularities (mainly) topologically identified. The second parts,* (SId) *and* (WId)*, are* identities*: we expect their validity under some additional analytic assumption. (In the last subsection we will verify all of them for rational singularities, — without any additional assumption.)*

4.4.2. Property (strong version). *Let $(X,0)$ be a normal surface singularity whose link M is a rational homology sphere. Let $\pi:\tilde{X}\to X$ be a fixed good resolution and $s:=\#\mathcal{J}$ the number of irreducible exceptional divisors of π. Consider an arbitrary $l'\in\mathbb{L}'$ and define the characteristic element $k:=K-2l'\in Char$. Then, we say that $(X,0)$ satisfies (SIn) or (SId) if:*

(SIn) for any line bundle $\mathcal{L}\in Pic(\tilde{X})$ with $c_1(\mathcal{L})=l'$ one has

$$h^1(\mathcal{L})\leq-\mathbf{sw}(M,[k])-\frac{k^2+s}{8}. \tag{1}$$

(SId) for $\mathcal{L}=\mathcal{O}_{\tilde{X}}(l')$ in (1) one has equality.

(For a generalization of *(SIn)* see 4.4.3(d), where the restriction $l'\in\mathbb{L}'$ is dropped.)

4.4.3. Remarks.

(a) If $\mathcal{L}=\mathcal{O}_{\tilde{X}}$ then $h^1(\mathcal{O}_{\tilde{X}})$ is the geometric genus p_g of $(X,0)$. For detailed historical remarks and list of cases for which (SIn) and (SId) are valid, see [37, 38, 39, 34, 33, 26].

(b) Notice that $\mathbf{sw}(M,[k])$ depends only on the class $[k]$ of k. In particular, the right hand side of (1) consists of the 'periodical' term $-\mathbf{sw}(M,[k])$ and the 'quadratic' term $-(k^2+s)/8$.

(c) In order to prove the property, it is enough to verify it for line bundles $\mathcal{L}$ with $c_1(\mathcal{L})=l'$ of type $l'=l'_{ne}(h)$ (for some $h\in H$).

Indeed, write l' in the form $l'=l'_1+l$ where $l'_1=e(l')=l'_{ne}(l'+L)$ and $l\in L_e$. Let $RHS(l')$, resp. $RHS(l'_1)$, be the right hand side of (1) for l', resp. l'_1. Since $[K-2l']=[K-2l'_1]$, the Seiberg-Witten invariants are the same, hence

$$RHS(l')-RHS(l'_1)=\frac{-(K-2l')^2+(K-2l'_1)^2}{8}=-(l,l')-\chi(l).$$

This combined with 4.3.3(c) shows that the statements of 4.4.2 for $\mathcal{L}$ and $\mathcal{L}\otimes\mathcal{O}_{\tilde{X}}(-l)$ are equivalent.

(c') Consider any set of representatives $\{l'\}_{l'\in R}$ ($R\subset \mathbb{L}'$) of the classes H, i.e. $\{l'+L\}_{l'\in R}=H$. Then the comparison (c) can be also be done for any $\mathcal{L}$ with $l':=c_1(\mathcal{L})\in R$ and for $\mathcal{L}\otimes\mathcal{O}_{\tilde{X}}(-l)$. Therefore, the validity of the property 4.4.2 follows from the verification of (1) for line bundles $\mathcal{L}$ with $c_1(\mathcal{L})\in R$.

E.g., one can take $R=Q$, or $R=-Q$. The importance of $R=-Q$ is emphasized by 4.2.9. This fact is exploited in the second version of the Property.

(d) Similarly, if one verifies the inequality (1) for any $l'\in\mathbb{L}'$, then one gets automatically the inequality (1) for *any* $l'\in L'$, hence for any $\mathcal{L}\in Pic(\tilde{X})$. This statement follows by induction: if the inequality (1) is valid for some $\mathcal{L}$, then it is valid for $\mathcal{L}\otimes\mathcal{O}_{\tilde{X}}(-E_j)$ (for any $j\in\mathcal{J}$).

Indeed, using the exact sequence $0\to\mathcal{L}\otimes\mathcal{O}_{\tilde{X}}(-E_j)\to\mathcal{L}\to\mathcal{L}|_{E_j}\to 0$, one gets that $h^1(\mathcal{L}\otimes\mathcal{O}_{\tilde{X}}(-E_j))\leq h^1(\mathcal{L})+1+(c_1(\mathcal{L}),E_j)$. The proof ends with similar comparison as in (c).

The point is that if $l'\notin\mathbb{L}'$, then the inequality (1), in general, is not sharp (optimal).

Theorem 9.6 of [34] implies the following fact, emphasizing once again the importance of almost rational singularities (cf. §2.7):

4.4.4. Theorem. *Part (SIn) of the strong version of the Property 4.4.2 (hence 4.4.3(d) too) is true for any almost rational singularity.*

4.4.5. Discussion. (a) Let $(X_a,0)$ be the universal abelian cover of $(X,0)$ with its natural H-action. Obviously, if $p:\tilde{Z}\to X_a$ is a resolution of $(X_a,0)$, then $\tilde{Z}$ inherits a natural H-action. Recall that the geometric genus $p_g(X_a,0)$ of $(X_a,0)$ is $h^1(\tilde{Z},\mathcal{O}_{\tilde{Z}})$. But one can define much finer invariants: consider the eigenspace decomposition $\oplus_{\chi\in\hat{H}}H^1(\tilde{Z},\mathcal{O}_{\tilde{Z}})_\chi$ of $H^1(\tilde{Z},\mathcal{O}_{\tilde{Z}})$, and take

$$p_g(X_a,0)_\chi:=\dim_{\mathbb{C}}H^1(\tilde{Z},\mathcal{O}_{\tilde{Z}})_\chi\quad(\text{for any }\chi\in\hat{H}).$$

(b) Obviously, we can repeat the above definition for *any* (unbranched) abelian cover of (X,0) . More precisely, for any epimorphism $H\to K$ one can take the composed map $\pi_1(M)\to H\to K$ which defines a Galois K-covering $(X_K,0)\to$(X,0) of (X,0) (with $(X_K,0)$ normal). Similarly as above, one can define $p_g(X_K,0)_\chi$ for any $\chi\in\hat{K}$. But these invariants are not essentially new: all of them can be recovered from the corresponding invariants associated with the universal abelian cover. Indeed, consider $\chi\in\hat{H}$ via $\hat{K}\hookrightarrow\hat{H}$. Then $p_g(X_K,0)_\chi=p_g(X_a,0)_\chi$. In particular,

$$p_g(X_K,0)=\sum_{\chi\in\hat{K}}p_g(X_a,0)_\chi.$$

(c) In the above definition (part (a)), $p_g(X_a,0)_\chi$ is independent of the choice of $\tilde{Z}$, in particular one can take $\tilde{Z}$ considered in the proof of 4.2.4. Those facts, together with 4.2.9 show that

$$p_g(X_a,0)_{\theta(h)}=h^1(\mathcal{O}_{\tilde{X}}(-l'_e(h)))\quad(\text{for any }h\in H).$$

Since the set $\{-l'_e(h)\}_{h\in H}$ is a set of representatives (cf. 4.4.3(c')), the previous version 4.4.2, *restricted to the set of line bundles of type* $\mathcal{O}_{\tilde{X}}(l')$, is completely equivalent with the following.

4.4.6. Property (weak version). *Let (X,0) be a normal surface singularity whose link M is a rational homology sphere. Let $\pi:\tilde{X}\to X$ be a fixed good resolution and $s:=\#\mathcal{J}$ the number of irreducible exceptional divisors of π. For any $h\in H=H_1(M,\mathbb{Z})$ consider the character $\chi:=\theta(h)$ and the characteristic element $k:=K+2l'_e(h)\in Char$.*

(WIn) for any $h\in H$

$$p_g(X_a,0)_{\theta(h)}\leq-\mathbf{sw}(M,[k])-\frac{k^2+s}{8}.\qquad(2)$$

(WId) in (2) one has equality.

4.4.7. Remark. (I). Notice that *(WIn)* (for all h) and 2.3.4(1) imply

$$p_g(X_a,0) \leq -\lambda(M) - \sum_{l'\in Q} \frac{(K+2l')^2+s}{8}, \tag{3}$$

or

$$p_g(X_a,0) \leq -\lambda(M) - |H|\cdot\frac{K^2+s}{8} + \sum_{l'\in Q}\chi(l').$$

(II). Assume that for some singularity one can verify the inequalities *(WIn)* (see e.g. 4.4.4). Then equality in (3) implies 4.4.6 with equalities for all h, i.e. *(WId)*.

4.4.8. Example. Assume that (X,0) $=\{x^2+y^3+z^{12t+2}=0\}$. Here we assume that $t\geq 1$ (if $t=0$ then (X,0) is rational, a case which will be clarified in the next section.) The following invariants of $(X,0)$ can be computed using [37], section 6.

The minimal good resolution graph of M is star-shaped with three arms and (normalized) Seifert invariants (α,ω) equal to $(3,1),(3,1),(6t+1,2t)$, the self intersection of the central curve is -1, the orbifold euler number equals $-1/(18t+3)$, $K^2+s=2$, $\lambda(M)=-(24t+1)/12$, and $H=\mathbb{Z}_3$.

Consider the two arms corresponding to Seifert invariants $(3,1)$. Both of them contain only one vertex with self intersection -3. We denote them by E_1, respectively E_2. Then Q contains exactly three element. They are 0, $l_1':=(E_1+2E_2)/3$ and $l_2':=(2E_1+E_2)/3$. One can rewrite

$$\sum_{l'\in Q}\frac{(K+2l')^2+s}{8} = |H|\cdot\frac{K^2+s}{8} - \sum_{l'\in Q}\chi(l').$$

In our case, by an easy verification $\chi(l_1')=\chi(l_2')=1/3$. Therefore, the right hand side of (Wa) is $(24t+1)/12-3\cdot 2/8+2/3=2t$.

On the other hand, the universal abelian cover of (X,0) is isomorphic to a Brieskorn singularity of type $(X_a,0)=\{u^3+v^3+w^{6t+1}=0\}$ (with the action $\xi*(u,v,w)=(\xi u,\bar{\xi}v,w)$, $\xi^3=1$). And one can verify easily that $p_g(X_a,0)=2t$. Hence, for (X,0) , (2) is valid with equalities.

In fact, in this case $p_g(X,0)=2t$ as well, hence $p_g(X_a,0)_\chi=0$ for any $\chi\neq 1$.

4.5 Example. The case of rational singularities.

4.5.1. Assume that $(X,0)$ is rational. Then by 3.4.1 and 3.5.2 one has

$$-\mathbf{sw}(M,[k]) = (k_r^2+s)/8. \tag{1}$$

4.5.2. Theorem. *Property 4.4.2 (hence 4.4.6 too) is true (with equality) for any rational singularity.*

Proof. By 4.4.3(c), we can assume that $l'=l'_{ne}(h)$ where $h=l'+L$. Using 4.3.2, and the fact that $l'_{ne}(h)\in L_{\mathbb{Q},ne}$, one gets that $h^1(\mathcal{O}_{\tilde{X}}(l'))=0$. On the other hand, by (1), $-\mathbf{sw}(M,[k])=(k_r^2+s)/8$. Here (see also 2.2.7), $k=K-2l'$, hence $k_r=K+2\bar{l}'_{ne}(-l'+L)$ with $\bar{l}'_{ne}(-l'+L)=-l'_{ne}(-(-l'+L))=-l'_{ne}(l'+L)=-l'$. Hence $k_r=k$ and the right hand side of 4.4.2(1) is also vanishing. □

Theorem 4.5.2 (and its proof) and the identity 4.5.1(1) have the following consequence (whose statement is independent of the Seiberg-Witten theory):

4.5.3. Corollary. *Assume that (X,0) is rational, and fix some $h\in H$. Then*

$$p_g(X_a,0)_{\theta(h)} = \frac{(K+2\bar{l}'_{ne}(h))^2-(K+2l'_e(h))^2}{8} = -\chi(\bar{l}'_{ne}(h))+\chi(l'_e(h)).$$

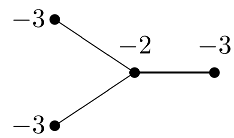
In particular, $(X_a,0)$ is rational if and only if $\chi(\bar{l}'_{ne}(h))=\chi(l'_e(h))$ for any $h\in H$.

This emphasizes in an impressive way the differences between the two 'liftings' $l'_e(h)$ and $\bar{l}'_{ne}(h)$. Recall: for a class $h=l'+L$, both $l'_e(h)$ and $\bar{l}'_{ne}(h)$ are elements of $l'+L$, but the first is minimal in

$L_{\mathbb{Q},e}$ (i.e. it is the representative in Q), while the second is minimal in $-L_{\mathbb{Q},ne}$. Since $-L_{\mathbb{Q},ne} \subset L_{\mathbb{Q},e}$, one has $l'_e(h) \leq \bar{l}'_{ne}(h)$. In some cases they are not equal.

For example, take the A_4 singularity, where E has three components E_1, E_2, E_3, all with self-intersection -2, and E_2 intersecting the others transversely. Then $-D_2 = (1/2, 1, 1/2) = \bar{l}'_{ne}(h)$ for some h, but it is not in Q: $l'_e(h) = -D_2 - E_2 = (1/2, 0, 1/2)$. Nevertheless, in this case, their Euler characteristics are the same (corresponding to the fact that — A_4 being a cyclic quotient singularity — the universal abelian cover is smooth).

4.5.4. Example. Assume that $(X, 0)$ is a rational singularity with the following dual resolution graph.



Let $\tilde{X}$ be its minimal resolution, we write E_0 for the central exceptional component, the others are denoted by E_i, $i = 1, 2, 3$. Then by a computation $l' := -(D_1 + D_2 + D_3) = (1, 2/3, 2/3, 2/3)$. This element is minimal in $-L_{\mathbb{Q},ne}$ but not in $L_{\mathbb{Q},e}$. Its representative in Q is $l' - E_0$. The character $\chi = \theta([l'])$ is $\chi(g_0) = 1$ and $\chi(g_j) = e^{4\pi i/3}$ for $j = 1, 2, 3$. By a computation $K = -l'$, $(K+2l')^2 = -2$ and $(K+2(l'-E_0))^2 = -10$. Therefore, $p_g(X_a, 0)_\chi = (-2+10)/8 = 1$. In particular, $(X_a, 0)$ cannot be rational.

Indeed, one can verify (using e.g. [40]), that the minimal resolution of $(X_a, 0)$ contains exactly one irreducible exceptional curve of genus 1 and self-intersection -3. In particular, $(X_a, 0)$ is minimally elliptic with $p_g(X_a, 0) = 1$. This also shows that the above eigenspace is the only non-trivial one.

We reverify this last fact for the conjugate $\bar{\chi}$ of χ. In this case $h = \theta^{-1}(\bar{\chi})$ is the class of $-D_0 = (1, 1/3, 1/3, 1/3) = \bar{l}'_{ne}(h)$; and $l'_e(h) = -D_0 - E_0$. By a calculation $K + 2\bar{l}'_{ne}(h) = E_0$ and $K + 2l'_e(h) = -E_0$, hence their squares are the same. In particular, $p_g(X_a, 0)_{\bar{\chi}} = 0$.

5 The graded roots of $S^3_{-p/q}(K)$.

5.1 Introduction.

In this section we determine the graded roots (in particular, the Heegaard Floer homology $HF^+(-M)$ and the Seiberg-Witten invariants $\mathbf{sw}^{OSZ}(M, \sigma)$) for the oriented 3-manifold $M = S^3_{-p/q}(K)$ obtained by a negative rational surgery (with coefficient $-p/q$) along an algebraic knot $K \subset S^3$. In this case, since $H_1(M, \mathbb{Z}) = \mathbb{Z}_p$, the *spin*^{c}-structures $\{\sigma_a\}_a$ of M can be parametrized by integers $a = 0, 1, \ldots p-1$. The main result establishes the graded roots in terms of the integers p, q, a, and the Alexander polynomial Δ of $K \subset S^3$. Notice that the Alexander polynomial of an algebraic (any) knot is well-understood, it can be easily computed from most of the other invariants of the knot (e.g., in the present algebraic case, from Puiseux or Newton pairs, or from the semigroup associated with the corresponding local analytic germ). In particular, the input of the theorem is the simplest that one can hope for. Since (in some sense) all the coefficients of Δ are effectively involved, in fact, the result is optimal.

In the very recent manuscript [53], Ozsváth and Szabó computed $HF^+(S^3_p(K))$ — for any knot K and any integer surgery coefficient p — in terms of the filtered chain homotopy type of the Heegaard Floer complex associated with the pair (S^3, K). Compared with this, our starting data, and also the description of $HF^+(-M)$, are simpler, and totally elementary; as a price for this we have to impose the 'negativity restrictions' for the surgery coefficient and for K.

The proof is based on the results and constructions of [34] valid for AR graphs. The needed facts are listed here in §3.5. Although [34], or §3.5 here, presents a precise algorithm how one should compute HF^+, its implementations in different situations sometimes is not straightforward. In the present case too, the proof and additional constructions run over two subsections. In fact, with this presentation, we also wish to advertise the efficiency, novelty and the power of [34]. (For another example, see [34], § 11, where the case of Seifert manifolds is treated.)

Subsection 2 also recalls the classical invariants of algebraic knots and connects them with the plumbing of M. The last subsection contains some concrete examples as well.

5.2 The manifold $S^3_{-p/q}(K_f)$.

5.2.1. Review of algebraic knots. [5, 7, 14] Let $K_f \subset S^3$ be the link of an *irreducible complex plane curve singularity* $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$; i.e. for $\epsilon > 0$ sufficiently small, write $S^3 = \{z \in \mathbb{C}^2 : ||z|| = \epsilon\}$ and take the transversal intersection $K_f := \{f = 0\} \cap S^3 \hookrightarrow S^3$. The natural orientations of S^3 and of the regular part of $\{f = 0\}$ induces a natural orientation on K_f. Since f is irreducible, $K_f \approx S^1$. We will assume that $\{f = 0\}$ is not smooth at the origin, i.e. $K_f \subset S^3$ is not the unknot. The isotopy type of $K_f \subset S^3$ is completely characterized by any of the following invariants listed below.

• The *Newton pairs* of f consist of $g \geq 1$ pairs of integers $\{(p_i, q_i)\}_{i=1}^g$, where $p_i \geq 2$, $q_i \geq 1$, $q_1 > p_1$ and $\gcd(p_i, q_i) = 1$.

• In some topological constructions, it is preferable to replace the Newton pairs by the *'linking pairs'* (or, the decorations of the splice diagram, cf. [7]) $(p_i, a_i)_{i=1}^g$, where

$$a_1 = q_1 \quad \text{and} \quad a_{i+1} = q_{i+1} + p_{i+1}p_i a_i \quad \text{for} \quad i \geq 1.$$

• Let $\Delta(t)$ be the *Alexander polynomial* of $K_f \subset S^3$, or equivalently, the *characteristic polynomial* of the algebraic monodromy acting on the first homology of the Milnor fiber of f. It is normalized by $\Delta(1) = 1$. In terms of the $(p_i, a_i)_i$ pairs it is

$$\Delta(t) = \frac{(t^{a_1p_1p_2\cdots p_g} - 1)(t^{a_2p_2\cdots p_g} - 1)\cdots(t^{a_gp_g} - 1)(t - 1)}{(t^{a_1p_2\cdots p_g} - 1)(t^{a_2p_3\cdots p_g} - 1)\cdots(t^{a_g} - 1)(t^{p_1\cdots p_g} - 1)}. \tag{1}$$

The degree of Δ, or equivalently, the first Betti number of the Milnor fiber of f, is the Milnor number μ of f.

• The *semigroup* S of the germ f is a sub-semigroup of $\mathbb{N}$ defined by $S := \{i_0(f, h) : h \in \mathcal{O}_{(\mathbb{C}^2,0)}\}$, where $i_0(f, h)$ denotes the local intersection multiplicity of f and h (which equals the codimension of the ideal (f, h) in $\mathcal{O}_{(\mathbb{C}^2,0)}$). For h invertible $i_0(f, h)$ is zero, hence $0 \in S$.

It is known that S is generated by the integers $\bar\beta_0 = p_1p_2\cdots p_g$, $\bar\beta_k = a_kp_{k+1}\cdots p_g$ for $1 \leq k \leq g-1$, and $\bar\beta_g = a_g$. Moreover, $\#(\mathbb{N} \setminus S)$ is finite. Its cardinality is the *delta-invariant* δ of f, which in this case also equals $\mu/2$, cf. [28]. It is also known that $\delta = \mu/2$ equals the minimal Seifert genus of $K_f \subset S^3$.

The largest element of $\mathbb{N} \setminus S$ is $\mu - 1$. In fact, for $0 \leq k \leq \mu - 1$ one has: $k \in S$ if and only if $\mu - 1 - k \notin S$.

• The *embedded minimal good dual resolution graph of the germ f* has the following shape

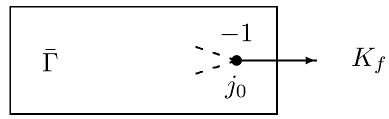
$\Gamma(f)$:

Above we emphasize only the vertices of degree one and three. The dash-line between two such vertices replaces a string  . The number of vertices of degree three is exactly g. In general, any vertex j is decorated by the self-intersection of the corresponding irreducible exceptional divisor E_j. In the above diagram we put only the decoration of the vertex j_0, which corresponds to the *unique* (-1)-curve, and which also supports the strict transform of $\{f = 0\}$. The other decorations are not really essential in our next discussions; for the description of the complete graph see e.g. [5], or [31], section 4.I.

The above diagram can also be identified with the plumbing graph of (S^3, K_f). In this case the self-intersections are the corresponding Euler numbers of the S^1-bundles, all the surfaces used in the plumbing construction are S^2's, and K_f is a generic fiber of the unique (-1)-bundle.

The above resolution graph $\Gamma(f)$ will be denoted by the following schematic diagram:

$\Gamma(f)$:



The polynomial $\Delta(t)$ can also be deduced from $\Gamma(f)$ by A'Campo's formula [1]. Notice that (if we disregard the arrowhead) the graph $\bar{\Gamma}$ can be blown down completely.

We emphasize again, the information codified in the following objects – isotopy class of $K_f \subset S^3$, set of Newton pairs, set of linking pairs, Alexander polynomial, semigroup or the embedded resolution graph – are completely equivalent. This means that the polynomial Q introduced below can be deduced from any of them.

5.2.2. Definition of the polynomial Q. One has the following identity connecting the Alexander polynomial Δ and the semigroup S (probably know already by Zariski and Teissier, see [14]):

$$\frac{\Delta(t)}{1-t} = \sum_{k \in S} t^k. \tag{2}$$

Since $\Delta(1) = 1$ and $\Delta'(1) = \delta$ (use e.g. the formula (1) of 5.2.1), one gets

$$\Delta(t) = 1 + \delta(t-1) + (t-1)^2 \cdot Q(t)$$

for some polynomial $Q(t) = \sum_{i=0}^{\mu-2} \alpha_i t^i$ of degree $\mu - 2$ with integral coefficients. In fact, all the coefficients $\{\alpha_i\}_{i=0}^{\mu-2}$ are strict positive, and:

$$\delta = \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{\mu-2} = 1.$$

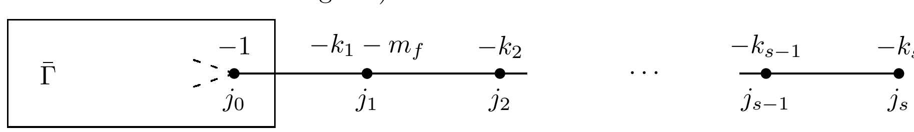
Indeed, by the above identity (2), one has $\delta + (t-1)Q(t) = \sum_{k \notin S} t^k$, or $Q(t) = \sum_{k \notin S}(t^{k-1} + \cdots + t + 1)$. This shows that

$$\alpha_i = \#\{k \notin S \ : \ k > i\}.$$

5.2.3. The surgery 3-manifold $M = S^3_{-p/q}(K)$. In the sequel we fix an oriented algebraic knot $K_f \subset S^3$ and we consider the oriented manifold $M := S^3_{-p/q}(K_f)$ obtained by $-p/q$-surgery along $K_f \subset S^3$, where $p/q > 0$ is a positive rational number ($p > 0$, $\gcd(p, q) = 1$).

It is easy to see (e.g. by Kirby calculus, see e.g. [11]) that M can be represented by the following plumbing diagram (the symbols $j_0, \ldots, j_s$ are the 'names' of the corresponding vertices, they are not really parts of the decoration of the diagram):

$\Gamma(M)$:



where $k_1 \geq 1$ and $k_j \geq 2$ ($2 \leq j \leq s$) are integers determined by the continued fraction $p/q = [k_1, k_2, \ldots, k_s]$ and $m_f = a_g p_g$. In terms of the embedded resolution graph of f (cf. 5.2.1), m_f is the multiplicity of the pull back of the germ f along the (-1)-curve. Topologically, e.g. if one uses Kirby calculus and blows down all the vertices of $\bar{\Gamma}$, m_f is a sum $\sum_i m_i^2$ of squares of a sequence of linking numbers m_i. In fact, in the language of plane curve singularities, this sequence of m_i's is exactly the 'multiplicity sequence' of f.

Notice that if we would start with the unknot $K_f \subset S^3$, then $M = S^3_{-p/q}(K_f)$ would be the lens space $L(p, q)$ (in general, normalized with $0 < q < p$, hence $k_1 \geq 2$ as well). This case is completely solved in § 3.6. Therefore, in the sequel we will assume that K_f is *not* the unknot; and we will allow the case $0 < p/q \leq 1$ (i.e. $k_1 = 1$) as well.

Take a point $* \in S^3 \setminus K_f$ and identify $S^3 \setminus *$ with $\mathbb{R}^3$. Then a Kirby diagram of M is given by the knot $K_f \subset \mathbb{R}^3$ with decoration (surgery coefficient) $-p/q$. In particular, the Kirby diagram of the manifold $-M$ (M with reversed orientation) — whose Heegaard Floer homology will be computed in the sequel — is given by the mirror image K_f^m of K_f (with respect to any plane in $\mathbb{R}^3$) with surgery coefficient p/q; i.e. $-M = S^3_{p/q}(K_f^m)$.

5.2.4. Lemma. *$\Gamma(M)$ is an AR-graph.*

Proof. We will show that if we modify the -1 decoration of j_0 in $\Gamma(M)$ into -2 (let us call this new graph by $\Gamma(M)_{-2}$), we get a sandwiched graph (cf. 2.5.3(d)). Indeed, consider the graph $\bar{\Gamma}$. Blow up the -1 vertex j_0. The new decoration of j_0 will be -2, while a new -1 vertex is created. Then blow up this new vertex $(m_f + k_1 - 1)$ times. Then its new decoration will be $-m_f - k_1$, while it has $(m_f + k_1 - 1)$ neighbours which are all -1 curves. Fix one of them, and blow up $k_2 - 1$ times. If one continues this procedure, one gets a graph which contains $\Gamma(M)_{-2}$ as a subgraph. □

5.2.5. Remark. Using 3.4.1, the above proof also shows that the plumbed 3-manifold associated with $\Gamma(M)_{-2}$ has trivial reduced Heegaard–Floer homology (i.e., it is an L-space in the sense of Ozsváth-Szabó).

5.3 The main invariants of $S^3_{-p/q}(K_f)$.

5.3.1. Consider $M = S^3_{-p/q}(K_f)$ as above. Then $H := H_1(M, \mathbb{Z}) = \mathbb{Z}_p$. There is natural identification of the *spin*^{c}-structures of M with the classes of $\mathbb{Z}_p$ – or, with the integers $a = 0, 1, \ldots, p-1$, $a = 0$ corresponding to the canonical *spin*^{c}-structure. For the precise identification, see 2.2.6 and 5.4.7. Let σ_a be the *spin*^{c}-structure associated with a.

The next theorem provides a purely combinatorial description of $HF^+(-M)$ from $\Gamma(M)$. More precisely, $HF^+(-M, \sigma_a)$ will be expressed in terms of the delta-invariant δ, the coefficients $\{\alpha_i\}_i$ of the polynomial Q (cf. 5.2.2), and the integers p, q and a. The symbol $\boldsymbol{s}(q,p)$ denotes the Dedekind sum (see e.g. 3.6). The integer q' is uniquely determined by $1 \le q' \le p$ and $qq' \equiv 1$ (*mod* p).

5.3.2. Theorem. *For each fixed $a = 0, 1, \ldots, p-1$, — corresponding to the p different spin^c-structures of M — one defines the following objects:*

- $t_a := \left[\frac{(2\delta-1)q-a-1}{p}\right]$;
- $r_a := 3\boldsymbol{s}(q,p) + 2\sum_{j=1}^{a}\left\{\frac{jq'}{p}\right\} - \frac{(1+2a)(p-1)}{2p} + \delta\Big(1 - \frac{q+1}{p}\Big) + \frac{\delta^2 q}{p} - \frac{2\delta a}{p}$;
- *a function $\tau_a : \{0, 1, \ldots, 2t_a + 2\} \to \mathbb{Z}$ by*

$$\begin{cases} \tau_a(2t) = t(1-\delta) + \sum_{i=0}^{t-1}\left[\frac{ip+a}{q}\right], & (t = 0, \ldots, t_a + 1); \\ \tau_a(2t+1) = \tau_a(2t+2) + \alpha_{[(tp+a)/q]}, & (t = 0, \ldots, t_a). \end{cases}$$

- *and the graded root $(R_{\tau_a}, \chi_{\tau_a})$ associated with τ_a.*

Then $(R_{\tau_a}, \chi_{\tau_a})$ is the graded root of M associated with σ_a. Hence the following facts also hold:

$$HF^+_{odd}(-M, \sigma_a) = 0, \quad HF^+_{even}(-M, \sigma_a) = \mathbb{H}(R_{\tau_a}, \chi_{\tau_a})[r_a], \quad \text{and} \quad -(k_r^2 + \#\mathcal{J})/4 = r_a;$$

$$d(-M, \sigma_a) = 2 \cdot \min \tau_a + r_a, \quad \text{where} \quad \min \tau_a = \tau_a(\, 2\lceil t_a/2 \rceil\,).$$

The proof is based on the theorems 3.5.2 and 3.5.6. The details will be provided in the subsections 5.4 and 5.5. We continue with some remarks and corollaries.

5.3.3. Remark. Notice that for any $t \in \{0, \ldots, t_a\}$, $\alpha_{[(tp+a)/q]}$ is strict positive, hence $\tau_a(2t+1) > \tau_a(2t+2)$. On the other hand, using properties of S (see e.g. 5.2.1, or 5.5.9(2)), one can verify that the following identity also holds:

$$\tau_a(2t+1) = \tau_a(2t) + \#\{\gamma \in S \,:\, \gamma \le\, (tp+a)/q\,\}.$$

Therefore, since $0 \in S$, one has $\tau_a(2t+1) > \tau_a(2t)$. In particular, the above representation of the graded root is the most 'economical': all the values are essential, see also 3.1.3(2).

This also shows that $(R_{\tau_a}, \chi_{\tau_a})$ has exactly $t_a + 2$ local minimum points, and they correspond to the values $\tau_a(2t)$, $t = 0, 1, \ldots, t_a + 1$.

5.3.2, 5.3.3 and 3.2.6 imply the next two corollaries.

5.3.4. Corollary.

$$\mathbf{sw}^{OSz}(M,\sigma_a) = -\mathbf{sw}^{OSz}(-M,\sigma_a) = \frac{r_a}{2} - \sum_{t\geq 0} \alpha_{[(tp+a)/q]}.$$

Recall that $HF^+(-M,\sigma_a)$ is a $\mathbb{Z}[U]$-module. Denote by $\ker U(\sigma_a)$ (resp. by $\operatorname{coker} U(\sigma_a)$) the kernel (resp. the cokernel) of the U-action. They are finitely generated graded free $\mathbb{Z}$-module. Let $\mathbb{Z}_{(r)}$ denote a rank one free $\mathbb{Z}$-module whose grading is r.

5.3.5. Corollary.

$$\ker U(\sigma_a) = \bigoplus_{0\leq t\leq t_a+1} \mathbb{Z}_{(2\tau_a(2t)+r_a)}.$$

In particular, $\ker U(\sigma_a)$ *depends only on the integers* p, q a, *and* δ, *but not on the coefficients* $\{\alpha_\ell\}_\ell$. *On the other hand,*

$$\mathit{coker}\, U(\sigma_a) = \bigoplus_{0\leq t\leq t_a} \mathbb{Z}_{(2\tau_a(2t+1)+r_a-2)},$$

which depends essentially on the coefficients $\{\alpha_\ell\}_\ell$ *of* Q. *Moreover:*

$$\operatorname{rank}_{\mathbb{Z}} \ker U = \sum_{a=0}^{p-1}(t_a+2) = p + (2\delta-1)q = p + \operatorname{rank}_{\mathbb{Z}} \operatorname{coker} U.$$

5.3.6. Remark. If $a \geq (2\delta-1)q$, then $t_a = -1$, hence $R_{\tau_a} = R_0$ (cf. 3.1.3(1)). In particular, for such a, $\mathbb{H}_{red}(R_{\tau_a}) = 0$. For all the other a's, $\mathbb{H}_{red}(R_{\tau_a})$ is not trivial. E.g., for the canonical *spin*^{c}-structure corresponding to $a=0$, $\mathbb{H}_{red}(R_{\tau_0})$ is never trivial. In particular, $-M$ is never an L-space.

5.4 The first part of the proof of 5.3.2: $k_r^2 + \#\mathcal{J}$

5.4.1. The index set $\mathcal{J}$. According to the shape of the plumbing graph $\Gamma(M)$, the index set of its vertices is $\mathcal{J} = \bar{\mathcal{J}} \cup \{1,\dots,s\}$, where $\bar{\mathcal{J}}$ is the index set of the vertices of $\bar\Gamma$, and the indices $\{1,\dots,s\}$ correspond to $j_1,\dots,j_s$. The distinguish vertex of $\Gamma(M)$ is j_0 corresponding to $0\in\bar{\mathcal{J}}\subset\mathcal{J}$. The base elements will also be denoted accordingly: $E_0, E_1,\dots,E_s$; or E_j for $j\in\bar{\mathcal{J}}$.

5.4.2. The graph ${}_\delta\Gamma$. As we already mentioned, the sub-graph $\bar\Gamma$ can be blown down. After this blow down, we get a graph which will be denoted as follows:

${}_\delta\Gamma$:

We can think about this graph as the dual graph of a minimal resolution (of a normal surface singularity) obtained from a resolution with dual graph $\Gamma(M)$. In this sense, the new vertex j_1 is a rational curve with a singular point which has delta-invariant δ; and the decorations $-k_i$ are the corresponding self-intersections. On the other hand, one can think about this graph also in the language of Kirby diagrams: j_1 represents the knot $K_f\subset S^3$, the other vertices represent unknots, they are linked as usual with linking number one, and $-k_i$ are the corresponding surgery coefficients.

The point is that a large number of numerical invariants of the graph $\Gamma(M)$ can already be determined from ${}_\delta\Gamma$ in terms of δ and $\{k_i\}_i$ (for this comparison, see the second part 5.4.6 of this subsection). The advantage of this is that the above graph ${}_\delta\Gamma$ is the same as the graph of a lens space $L(p,q)$, provided that we disregard the decoration $[\delta]$. In particular, its invariants can be computed by similar methods as those of lens spaces – in fact, in their computations we will even use the corresponding relations valid for lens spaces. This will be the subject of the first part of this subsection. Our model is subsection 3.6 where we run the algorithm for lens spaces. The reader is invited to consult this subsection and also the original source [34] and to verify that our present claims, verified in 3.6 for case $q < p$, are valid for $q \geq p$ as well.

In fact, any invariant of the lattice (which does not involve δ) equals the corresponding invariant of $L(p, q)$. But, formulas which involve δ (like the 'adjunction formula' determining the canonical characteristic element, or the 'Riemann-Roch' formula for χ_k) depend essentially on δ.

5.4.3. Notations. In order to make a distinction between invariants of the graphs $\Gamma(M)$ and ${}_\delta\Gamma$, the invariants of the second one will have an extra $\tilde{}$-decoration; e.g. $\tilde{L}$ denotes the lattice of rank s with base elements $\{\tilde{E}_i\}_{i=1}^s$, while $\{\tilde{D}_i\}_{i=1}^s \in \tilde{L}'$ are the dual bases, etc.

The integers $\{n_{is}\}_{i=1}^s$ and have the same meaning as in 3.6.1. Similarly, q' is the unique integer with $p \geq q' \geq 1$ and $qq' \equiv 1$ ($mod\ p$) (notice the small difference with 3.6.1 where $q < p$, but in the present case $p < q$ is allowed too).

5.4.4. The group $\tilde{H}$ and the elements $\tilde{l}'_{[k]}$. Clearly, cf. 3.6, $\tilde{H} = \tilde{L}'/\tilde{L} = \mathbb{Z}_p$, and $[\tilde{D}_s] = \tilde{D}_s + \tilde{L}$ is one of its generators. Then $[\tilde{D}_j] = [n_{j+1,s}\tilde{D}_s]$ in $\tilde{H}$ $(1 \leq j \leq s)$. The set of $spin^c$-structures is the set of orbits $\{-a\tilde{D}_s + \tilde{L}\}_{0 \leq a < p}$. For any $0 \leq a < p$ write

$$\tilde{l}'_{[-a\tilde{D}_s]} = -(a_1\tilde{D}_1 + a_2\tilde{D}_2 + \cdots + a_s\tilde{D}_s).$$

By 3.6.2, the system $(a_1, \ldots, a_s)$ can be realized as the set of coefficients of a minimal vector $\tilde{l}'_{[-a\tilde{D}_s]}$ for some $0 \leq a < p$ if and only if the entries satisfy the system of inequalities (SI) of 3.6.2.

5.4.5. The canonical characteristic element. Let ${}_\delta\tilde{K}$ denote the canonical characteristic element associated with the graph ${}_\delta\Gamma$ (see below). It is convenient to consider the canonical characteristic element $\tilde{K}$ of the graph of $L(p, q)$ (recall that this graph is obtained from ${}_\delta\Gamma$ by omitting δ). The adjunction relations defining $\tilde{K}$ are the usual ones, but those which identify ${}_\delta\tilde{K}$ are the following (see also 5.4.9): $({}_\delta\tilde{K}, \tilde{E}_j) - k_j + 2$ should equal twice the delta-invariant of the vertex j, namely $= 2\delta$ for $j = 1$, and $= 0$ otherwise. Hence:

$${}_\delta\tilde{K} = \tilde{K} + 2\delta\tilde{D}_1.$$

In particular,

$${}_\delta\tilde{K}^2 + s = \tilde{K}^2 + s + 4\delta(\tilde{K}, \tilde{D}_1) + 4\delta^2(\tilde{D}_1, \tilde{D}_1).$$

$(\tilde{K}, \tilde{D}_1)$ is the $\tilde{E}_1$-coefficient of $\tilde{K}$, which equals $-1 + (q + 1)/p$, see e.g. [37], (5.2). Similarly, $\tilde{D}_1^2 = \tilde{I}_{11}^{-1} = -n_{2s}/p = -q/p$. Also, by 3.6.5:

$$\tilde{K}^2 + s = 2(p - 1)/p - 12 \cdot \boldsymbol{s}(q, p).$$

Therefore,

$${}_\delta\tilde{K}^2 + s = \frac{2(p-1)}{p} - 12\boldsymbol{s}(q, p) - 4\delta(1 - \frac{q+1}{p}) - 4\delta^2\frac{q}{p}.$$

The distinguished characteristic element $\tilde{k}_r$ of the orbit $-a\tilde{D}_s + \tilde{L}$ is

$$\tilde{k}_r = {}_\delta\tilde{K} + 2\tilde{l}'_{[-a\tilde{D}_s]}.$$

Therefore,

$$\tilde{k}_r^2 + s = {}_\delta\tilde{K}^2 + s + 4 \cdot (\tilde{K} + \tilde{l}'_{[-a\tilde{D}_s]}, \tilde{l}'_{[-a\tilde{D}_s]}) + 8\delta \cdot (\tilde{D}_1, \tilde{l}'_{[-a\tilde{D}_s]}).$$

By 3.6.6

$$(\tilde{K} + \tilde{l}'_{[-a\tilde{D}_s]}, \tilde{l}'_{[-a\tilde{D}_s]}) = \frac{a(p-1)}{p} - 2 \cdot \sum_{j=1}^{a} \left\{\frac{jq'}{p}\right\}.$$

On the other hand, by formula (1) of 3.6.2:

$$(\tilde{D}_1, \tilde{l}'_{[-a\tilde{D}_s]}) = -\sum_{i=1}^{s} a_i(\tilde{D}_1, \tilde{D}_i) = \sum_{i=1}^{s} a_i\frac{n_{i+1,s}}{p} = \frac{a}{p}.$$

Summing all, one gets

$$-\frac{\tilde{k}_r^2+s}{4}=3\boldsymbol{s}(q,p)+2\sum_{j=1}^{a}\Big\{\frac{jq'}{p}\Big\}-\frac{(1+2a)(p-1)}{2p}+\delta\Big(1-\frac{q+1}{p}\Big)+\frac{\delta^2 q}{p}-\frac{2\delta a}{p}.$$

5.4.6. Back to the graph $\Gamma(M)$. Finally, we compare the invariants of the graphs $\Gamma(M)$ and ${}_\delta\Gamma$.

There are two natural morphisms connecting the corresponding lattices L and $\tilde{L}$. The first one, $\pi_*: L\to\tilde{L}$ is defined by $\pi_*(E_j)=0$ for any $j\in\bar{\mathcal{J}}$, while $\pi_*(E_i)=\tilde{E}_i$ for $i=1,\ldots,s$.

In order to define the second morphism, we need an additional construction.

Let $Z_f:=\sum_{j\in\bar{\mathcal{J}}}m_jE_j$ be the cycle supported by $\bar{\Gamma}$ which satisfies

$$(Z_f+E_1,E_j)=0 \quad \text{for } j\in\bar{\mathcal{J}}.$$

Since $\det\bar{\Gamma}=\pm 1$, this system has a unique integral solution $\{m_j\}_j$. In fact, in terms of the diagram $\Gamma(f)$ (cf. 5.2.1), m_j is the vanishing order of the pull back of f along the corresponding irreducible exceptional divisor. E.g., $m_0=m_f$.

Then one defines $\pi^*:\tilde{L}\to L$ by $\pi^*(\tilde{E}_j)=E_j$ for $j\geq 2$ and $\pi^*(\tilde{E}_1)=Z_f+E_1$. By the very definition follows the 'projection formula':

$$(\pi^*(\tilde{l}),l)=(\tilde{l},\pi_*(l)) \quad \text{for } \tilde{l}\in\tilde{L},\ l\in L.$$

5.4.7. The group H. π_* has a natural extension to $L_{\mathbb{Q}}\to\tilde{L}_{\mathbb{Q}}$, and $\pi_*(L')\subset\tilde{L}'$. Therefore, π_* induces a group morphism $H\to\tilde{H}$. Its surjectivity follows from $\pi_*\pi^*=1$. In order to prove injectivity, consider an $l'\in L'$ with $\pi_*(l')\in\tilde{L}$. Then $\pi^*\pi_*(l')$ is an element of L. Since $\pi^*\pi_*(l')-l'$ is supported by $\bar{\Gamma}$, and $\det\bar{\Gamma}=\pm1$, one gets that $\pi^*\pi_*(l')-l'\in L$ as well. Hence $l'\in L$, and π_* induces an isomorphism $H\to\tilde{H}=\mathbb{Z}_p$, and H is generated by D_s.

In the sequel, the set of *spin*^{c}-structures of M will be identified with the set of orbits $\{-aD_s+L\}_{0\leq a<p}$. We denote by σ_a that *spin*^{c}-structure which correspond to the orbit $-aD_s+L$.

5.4.8. The element $l'_{[-aD_s]}$. We claim that $l'_{[-aD_s]}=\pi^*(\tilde{l}'_{[-a\tilde{D}_s]})$. Indeed, $l':=\pi^*(\tilde{l}'_{[-a\tilde{D}_s]})\in(-L_{\mathbb{Q},ne})$ by the projection formula. Next, we verify the minimality of l'. Notice that any $l\in L$ with $l\geq 0$ can be written in the form $\pi^*(\tilde{x})+\bar{x}$, where $\tilde{x}\geq 0$ and $\bar{x}$ is supported by $\bar{\Gamma}$. Assume that for such $\pi^*(\tilde{x})+\bar{x}\geq 0$ one has:

$$(l'-\pi^*(\tilde{x})-\bar{x},E_j)\leq 0 \ \text{ for any } j\in\mathcal{J}. \qquad (*)$$

Since $(\pi^*(\tilde{y}),E_j)=0$ for any $j\in\bar{\mathcal{J}}$, one gets that $(-\bar{x},E_j)\leq 0$ for any $j\in\bar{\mathcal{J}}$. Since $\bar{\Gamma}$ is negative definite, it follows that $\bar{x}\leq 0$. Using this, $(*)$ for $j=1,\ldots,s$ gives

$$0\geq(l'-\pi^*(\tilde{x}),E_j)+(-\bar{x},E_j)\geq(l'-\pi^*(\tilde{x}),E_j)=(\tilde{l}'_{[-a\tilde{D}_s]}-\tilde{x},\tilde{E}_j).$$

hence, by the minimality of $\tilde{l}'_{[-a\tilde{D}_s]}$, one has $\tilde{x}=0$. But then $\bar{x}\geq 0$ and $\bar{x}\leq 0$ implies $\bar{x}=0$ as well.

5.4.9. Claim: $\pi_*(K)={}_\delta\tilde{K}$. First notice that by the projection formula:

$$\pi_*(D_j)=\begin{cases} m_j\tilde{D}_1 & \text{if } j\in\bar{\mathcal{J}}\\ \tilde{D}_j & \text{if } j=1,\ldots,s.\end{cases}$$

(where $Z_f=\sum_{j\in\bar{\mathcal{J}}}m_jE_j$ as above). The adjunction relations for $\Gamma(M)$ can be rewritten as

$$K=-\sum_{j\in\mathcal{J}}E_j-\sum_{j\in\mathcal{J}}(2-s_j)D_j,$$

where s_j is the adjacent degree of the vertex j in $\Gamma(M)$. They also satisfy (cf. [1]):

$$\sum_{j\in\bar{\mathcal{J}}}(2-s_j)m_j=1-\mu=1-2\delta.$$

Then, taking π_* one gets $\pi_*(K)=\tilde{K}+2\delta\tilde{D}_1={}_\delta\tilde{K}$.

5.4.10. Claim: $K^2 + \#\mathcal{J} = \ {}_\delta\tilde{K}^2 + s$**.** Indeed, set $K_{\bar\Gamma} := K - \pi^*({}_\delta\tilde{K})$. Then $K_{\bar\Gamma}$ is supported by $\bar\Gamma$ and satisfies the adjunction relations $(K_{\bar\Gamma}, E_j) = -e_j - 2$ for any $j \in \bar{\mathcal{J}}$, hence it is the canonical cycle of $\bar\Gamma$. Moreover, since $\bar\Gamma$ can be blown down (and since by an elementary blow up of a smooth point K^2 decreases by 1), one has $K_{\bar\Gamma}^2 = -(\#\mathcal{J} - s)$. By projection formula $\pi^*({}_\delta\tilde{K})$ is orthogonal to $K_{\bar\Gamma}$, hence

$$K^2 = (\pi^*({}_\delta\tilde{K}) + K_{\bar\Gamma})^2 = \ {}_\delta\tilde{K}^2 + K_{\bar\Gamma}^2 = \ {}_\delta\tilde{K}^2 - (\#\mathcal{J} - s).$$

Finally, this claim, via the projection formula, shows for any *spin*^{c}-structure that

$$k_r^2 + \#\mathcal{J} = \tilde{k}_r^2 + s.$$

5.4.11. Example. Integer surgery. Assume that $q = 1$, i.e. $M = S^3_{-p}(K_f)$. Then $q' = 1$ as well, and

$$-\frac{k_r^2 + \#\mathcal{J}}{4} = -\frac{\tilde{k}_r^2 + s}{4} = \frac{(p + 2\delta - 2 - 2a)^2 - p}{4p}.$$

5.4.12. Example. $(1/q)$**-surgery.** Assume that $p = 1$, i.e. $M = S^3_{-1/q}(K_f)$. Then again $q' = 1$. There is only one *spin*^{c}-structure corresponding to $a = 0$, and

$$-\frac{K^2 + \#\mathcal{J}}{4} = -\frac{\tilde{K}^2 + s}{4} = q\delta(\delta - 1).$$

5.5 The second part of the proof of 5.3.2: $(R_{\tau_{[k]}}, \chi_{\tau_{[k]}})$

The construction of the cycles $x_{[k]}(i)$ is given in two steps. The first step provides similar cycles associated with the graph $\bar\Gamma$, and it is based on the combinatorial properties of the graph $\Gamma(f)$ (involving also some techniques of plane curve singularities). In particular, we prefer to think about $\bar\Gamma$ as the dual graph of irreducible exceptional curves obtained by repeated blow ups of $(\mathbb{C}^2, 0)$. The lattice of $\bar\Gamma$ is denoted by $\bar{L}$, and we write $\bar{L}_{ne} := \{y \in \bar{L} : (y, E_j) \geq 0,\ j \in \bar{\mathcal{J}}\}$.

5.5.1. The cycles $\{y(i)\}_{i\geq 0}$**.** Since $\bar\Gamma$ is negative definite, (7.6) of [34] guarantees, for any $i \geq 0$, the existence of a positive cycle $y(i) \geq 0$ (supported by $\bar\Gamma$) with the following properties:
(a) $pr_0(y(i)) = i$;
(b) $(y(i), E_j) \leq 0$ for any $j \in \bar{\mathcal{J}} \setminus \{0\}$;
(c) $y(i)$ is minimal (with respect the partial ordering $\leq$) with the properties (a) and (b).

E.g., $y(0) = 0$. Although there is very precise algorithm which determines all the cycles $y(i)$ (see e.g. the proof of 5.5.3, or [34]), we are not interested in all the coefficients of $y(i)$. Instead, what we really have to know is the set of intersection numbers $(y(i), E_0)$ (cf. 3.5.7). Let Z_f be the divisorial part (supported by $\bar\Gamma$) of the germ f, cf. 5.4.6, which satisfies

$$(Z_f, E_j) = \begin{cases} 0 & \text{if } j \in \bar{\mathcal{J}} \setminus \{0\}; \\ -1 & \text{if } j = 0. \end{cases}$$

Recall also that $S \subset \mathbb{N}$ denotes the semigroup of f, see 5.2.1.

5.5.2. Proposition.
(a) If $i = tm_f + i_0$ with $t \geq 0$ and $0 \leq i_0 < m_f$, then $y(i) = tZ_f + y(i_0)$;
(b) For any $i < m_f$ one has

$$(y(i), E_0) = \begin{cases} 1 & \text{if } i \notin S; \\ 0 & \text{if } i \in S. \end{cases}$$

Proof. *(a)* Clearly, $y'(i) := tZ_f + y(i_0)$ satisfies (5.5.1)(a)-(b), we need to verify (c). But if for some $y \geq 0$, $y'(i) - y$ satisfies (a)-(b), then $y(i_0) - y$ would also satisfy (a)-(b) for i_0, hence $y = 0$ by minimality of $y(i_0)$. Part (b) of (5.5.2) will follow from the next sequence of lemmas. ◻

The first lemma was proved and used intensively in [34] as a general principle of rational graphs. For the convenience of the reader, we sketch its proof, for more details, see [loc. cit.].

5.5.3. Lemma. *$(y(i),E_0)\leq 1$ for any $i\geq 0$.*

Proof. We denote by Z_{min} Artin's minimal cycle associated with $\bar{\Gamma}$, i.e. the minimal nonzero cycle in $-\bar{L}_{ne}$. The cycle Z_{min} can be determined by the following (Laufer's) algorithm ([20]): Construct a 'computation' sequence of cycles $\{Z_l\}_l$ as follows. Set $Z_0:=0$, $Z_1:=b_0$; if Z_l is already constructed, but for some $j\in\bar{\mathcal{J}}$ one has $(Z_l,E_j)>0$, then take $Z_{l+1}:=Z_l+b_j$. If $Z_l\in(-\bar{L}_{ne})$ then stop and $Z_l=Z_{min}$.

Now, by a geometric genus computation, one gets for all rational graphs that whenever $(Z_l,E_j)>0$ in the above algorithm, in fact one has the equality (†): $(Z_l,E_j)=1$ (cf. [20]). Since $\bar{\Gamma}$ can be blown down, it is rational, hence (†) works.

The point is that $y(i)$ can also be computed by a similar algorithm: Assume that $y(i)$ is already determined. Then construct the sequence of cycles $\{Z_l\}_l$ as follows. $Z_0:=y(i)$, $Z_1:=y(i)+E_0$, and if (for $l\geq 1$) $(Z_l,E_j)>0$ for some $j\in\bar{\mathcal{J}}\setminus\{0\}$, then take $Z_{l+1}:=Z_l+E_j$, otherwise stop and write $Z_l=y(i+1)$.

Then one can verify that any sequence connecting $y(i)$ with $y(i+1)$ can be considered as part of a computation sequence associated with Z_{min}, hence lemma follows by the above property (†). □

5.5.4. Lemma. *Fix an arbitrary $i\geq 0$. Then $(y(i),E_0)\leq 0$ if and only if $i\in S$.*

Proof. '⇒' If $(y(i),E_0)\leq 0$ then $y(i)\in(-\bar{L}_{ne})$. Since $\bar{\Gamma}$ is the dual graph of a modification of $(\mathbb{C}^2,0)$, the cycle $y(i)$ is the divisorial part Z_g of a holomorphic germ $g\in\mathcal{O}_{(\mathbb{C}^2,0)}$. Since the intersection multiplicity $i_0(f,g)$ of the germs f and g at $0\in\mathbb{C}^2$ is the multiplicity of Z_g along the exceptional divisor supporting the arrow of f, one gets $i_0(f,g)=pr_0(y(i))=i$. But, by definition, $i_0(f,g)\in S$.

'⇐' If $i\in S$, then there exists $g\in\mathcal{O}_{(\mathbb{C}^2,0)}$ with $i_0(f,g)=i$; i.e. with $pr_0(Z_g)=i$ (see above) and $Z_g\in(-\bar{L}_{ne})$ (a general property of divisorial parts of holomorphic germs). Then the minimality of $y(i)$ implies $y(i)\leq Z_g$. This, and $pr_0(y(i))=pr_0(Z_g)$, and the fact that I off the diagonal has positive entries imply that $(E_0,y(i))\leq(E_0,Z_g)$. But, since $Z_g\in(-\bar{L}_{ne})$, $(E_0,Z_g)\leq 0$. □

Lemma 5.5.4 can be improved for $i<m_f$:

5.5.5. Lemma. *Assume that $i<m_f$. Then $(y(i),E_0)=0$ if and only if $i\in S$.*

Proof. Via 5.5.4, it is enough to prove that $(y(i),E_0)<0$ is not possible for $i<m_f$. Indeed, assume that $(y(i),E_0)<0$ for some i. Then, by a verification $y(i)-Z_f\in(-\bar{L}_{ne})$, hence (since $\bar{\Gamma}$ is negative definite) $y(i)-Z_f\geq 0$. In particular, $i-m_f=pr_0(y(i)-Z_f)\geq 0$. □

5.5.6. Back to $\Gamma(M)$. We consider again the graph $\Gamma(M)$. We fix an orbit $[k]=-aD_s+L$ for some $0\leq a<p$ (corresponding to the $spin^c$-structure σ_a of M). In the sequel we will use freely the notations of the previous subsection. Additionally, we write for any $j=1,\dots,s$:

$$a'_j:=n_{j+1,s}a_j+n_{j+2,s}a_{j+1}+\ldots+a_s.$$

E.g., by 3.6.2, $a'_1=a$.

Consider now the sub-graph of $\Gamma(M)$ with vertices $j_0,\dots,j_s$ and the s edges connecting them. We wish to identify the cycle $z(i)=iE_0+\sum_{i=1}^s u_jE_j$ which has the properties of $x(i)$ 'restricted on this sub-graph'. More precisely:

5.5.7. Proposition. *Fix $i\geq 0$ and $0\leq a<p$.*

(a) For $j=1,\dots,s$ define the integers u_j inductively by:

$$u_1:=\left\lceil\frac{iq-a}{p+qm_f}\right\rceil,\quad u_j:=\left\lceil\frac{u_{j-1}n_{j+1,s}-a'_j}{n_{j,s}}\right\rceil\ (1<j\leq s).$$

Then the cycle $z(i):=iE_0+\sum_{j=1}^s u_jE_j$ is ≥ 0 and satisfies

$$(z(i)+l'_{[-aD_s]},E_j)\leq 0\ \textit{for any}\ \ j=1,\dots,s.$$

(b) *Assume that the cycle* $\bar z(i) := iE_0 + \sum_{j=1}^s \bar u_j E_j$ *is* ≥ 0 *and satisfies*

$$(\bar z(i) + l'_{[-aD_s]}, E_j) \leq 0 \text{ for any } \ j = 1, \dots, s.$$

Then

$$\bar u_1 \geq \left\lceil \frac{iq-a}{p+qm_f} \right\rceil, \quad \text{and} \ \ \bar u_j \geq \left\lceil \frac{\bar u_{j-1} n_{j+1,s} - a'_j}{n_{j,s}} \right\rceil \text{ for } \ 1 < j \leq s.$$

Proof. This property was used in a similar situation for the plumbing graph of a Seifert manifolds, applied for its 'legs', cf. 11.11 of [34]. For the convenience of the reader we sketch the proof.

(a) By the definition of u_j one has $u_j n_{js} \geq u_{j-1} n_{j+1,s} - a'_j$. This is equivalent, via the identities $n_{js} = k_j n_{j+1,s} - n_{j+2,s}$, resp. $a'_j = n_{j+1,s} a_j + a'_{j+1}$, with

$$u_j k_j - u_{j-1} + a_j \geq \frac{u_j n_{j+2,s} - a'_{j+1}}{n_{j+1,s}}.$$

Using the definition of u_{j+1}, one gets $u_j k_j - u_{j-1} + a_j \geq u_{j+1}$, which is exactly $(z(i) + l'_{[-ag_s]}, E_j) \leq 0$. (b) can be proved by a descending induction, using the above sequence of arguments in reversed order. ◻

5.5.8. Corollary. *Fix* $0 \leq a < p$ *and write* $i = tm_f + i_0$ *for some* $t \geq 0$ *and* $0 \leq i_0 < m_f$. *Then*

$$x_{[k]}(i) = t \cdot Z_f + y(i_0) + \left\lceil \frac{iq-a}{qm_f+p} \right\rceil E_1 + \sum_{j \geq 2} u_j E_j.$$

In particular,

$$(x_{[k]}(i), E_0) = -t + \left\lceil \frac{iq-a}{qm_f+p} \right\rceil + (y(i_0), E_0).$$

Moreover:

$$\chi_{k_r}(\, x_{[k]}(i+1)\,) - \chi_{k_r}(\, x_{[k]}(i)\,) = t + 1 - \left\lceil \frac{iq-a}{qm_f+p} \right\rceil - \begin{cases} 1 & \text{if } i_0 \notin S \\ 0 & \text{if } i_0 \in S. \end{cases}$$

Proof. Since the subgraphs $\bar\Gamma$ and its complement are connected in $\Gamma(M)$ only through j_0, the two parts of $x_{[k]}(i)$ — one supported by $\bar\Gamma$, the other by its complement — have independent lives. But, for any $j \in \bar{\mathcal{J}}$, $(x_{[k]}(i) + l'_{[k]}, E_j) = (x_{[k]} + \pi^*(\tilde l'_{[k]}), E_j) = (x_{[k]}, E_j)$. Hence $x_{[k]}(i)$ restricted on $\bar\Gamma$ should be exactly $y(i)$. Clearly, the restriction on the complement of $\bar\Gamma$ should be $z_{[k]}(i)$. Hence the first two statements follow. The last is the consequence of 3.5.7 together with the identity $\chi_{k_r}(E_0) = 1$, for $(l'_{[k]}, E_0) = (\pi^*(\tilde l'_{[k]}), E_0) = 0$. ◻

5.5.9. τ_a**.** Corresponding to this and 3.5.6, we write $\tau_a(i) := \chi_{k_r}(x_{[k]}(i))$. With the notations of 5.5.8, notice that

$$\frac{iq-a}{qm_f+p} \leq t+1,$$

hence $\tau_a(i+1) - \tau_a(i) \geq -1$ for any i, and $= -1$ only in very special situations, namely if

$$\frac{(tm_f+i_0)q-a}{qm_f+p} > t \ \ \text{and} \quad i_0 \notin S. \tag{1}$$

In order to analyze when is this possible, we will consider sequences $Seq(t) := \{tm_f + i_0 : 0 \leq i_0 < m_f\}$ for fixed $t \geq 0$. In such a sequence, notice that the very last element of $\mathbb{N} \setminus S$, namely $\mu - 1 = 2\delta - 1$ is strict smaller than $m_f - 1$ (a fact, which can be proved by induction over the number of Newton pairs), hence the complete set $\mathbb{N} \setminus S$ sits in $\{0, \dots, m_f - 1\}$. Therefore, there exists in $Seq(t)$ an i_0 satisfying (1) if and only if

$$\frac{(tm_f + 2\delta - 1)q - a}{qm_f + p} > t.$$

This is equivalent to $t \leq t_a$, for t_a defined in 5.3.2. In other words, if $i \geq T_0 := (t_0 + 1)m_f$, then $\tau_a(i+1) \geq \tau_a(i)$, hence those values of τ_a provide no contribution in the graded root (cf. 3.1.3). Moreover, for $t \in \{0, \dots, t_a\}$, in $Seq(t)$ one has:

$$\Delta(i_0) := \tau_a(tm_f + i_0 + 1) - \tau_a(tm_f + i_0) = \begin{cases} 0 & \text{if } i_0 \leq (tp+a)/q, & \text{and } i_0 \notin S; \\ +1 & \text{if } i_0 \leq (tp+a)/q, & \text{and } i_0 \in S; \\ -1 & \text{if } i_0 > (tp+a)/q, & \text{and } i_0 \notin S; \\ 0 & \text{if } i_0 > (tp+a)/q, & \text{and } i_0 \in S. \end{cases}$$

In particular, $\Delta(i_0) \geq 0$ for $0 \leq i_0 \leq (tp+a)/q$ with exactly

$$A_t := \#\{\gamma \in S : \gamma \leq (tp+a)/q\}$$

times taking the value $+1$, otherwise zero; and $\Delta(i_0) \leq 0$ for $i_0 > (tp+a)/q$ with exactly

$$B_t := \#\{\gamma \notin S : \gamma > (tp+a)/q\}$$

times taking the value -1, otherwise zero. Notice that both A_t and B_t are strict positive (since $0 \in S$, respectively $2\delta - 1 \notin S$ and $2\delta - 1 > (tp+a)/q$). This shows that

$$M_t := \max_{0 \leq i_0 < m_f} \tau_a(tm_f + i_0) = \tau_a(tm_f) + A_t = \tau_a((t+1)m_f) + B_t, \tag{2}$$

and

$$M_t > \max\{\, \tau_a(tm_f), \tau_a(tm_f + m_f) \,\}.$$

Therefore, the graded root associated with the values $\{\tau_a(i)\}_{0 \leq i \leq (t_a+1)m_f}$ is the same as the graded root associated with the values

$$\tau_a(0), M_0, \tau_a(m_f), M_1, \tau_a(2m_f), M_2, \dots, \tau_a(t_a m_f), M_{t_a}, \tau_a(t_a m_f + m_f).$$

Finally notice, since $\#\{\gamma \notin S\} = \delta$, one has $\delta - B_t = \#\{\gamma \notin S : \gamma \leq (tp+a)/q\}$, hence $\delta - B_t + A_t = [(tp+a)/q] + 1$. Hence, by (2),

$$\tau_a((t+1)m_f) - \tau_a(tm_f) = \Big[\frac{tp+a}{q}\Big] + 1 - \delta.$$

Since $\tau_a(0) = 0$, this gives $\tau_a(tm_f)$ inductively. Notice also that $B_t = \alpha_{[(tp+a)/q]}$.

Clearly, the graded root associated with τ_a is the same as the graded root associated with $\tilde{\tau}_a : \{0, 1, 2, \dots, 2t_a + 2\} \to \mathbb{Z}$, where $\tilde{\tau}_a(2t) := \tau_a(tm_f)$ and $\tilde{\tau}_a(2t+1) := M_t$. This is the tau-function of 5.3.2 if we delete $\tilde{}$.

5.6 Examples

5.6.1. Example. Assume $p = q = 1$. In this case M is integral homology sphere; $a = 0$ and $t_0 = 2\delta - 2 = \mu - 2$. In particular, the rank of $\ker U$ is μ. Moreover, $r_0 = \delta(\delta - 1)$ and $\tau_0(2t) = t(t - 2\delta + 1)/2$. The reader is invited to draw the graded root and verify that

$$HF^+(-M) = \mathcal{T}_0^+ \oplus \mathcal{T}_0(\alpha_{\delta-1}) \oplus \bigoplus_{i=1}^{\delta-1} \mathcal{T}_{i(i+1)}(\,\alpha_{i-1+\delta}\,)^{\oplus 2};$$

$$\ker U = \bigoplus_{i=0}^{\delta-1} \mathbb{Z}^2_{(i^2+i)}.$$

5.6.2. Example. More generally, assume only that $p = 1$. The Heegaard Floer homology of the unique *spin*^{c}-structure is:

$$HF^+(-M) = \mathcal{T}_0^+ \oplus \mathcal{T}_0(\alpha_{\delta-1})^{\oplus q} \oplus \bigoplus_{i=1}^{(\delta-1)q} \mathcal{T}_{([l/q]+1)(\{l/q\}q+l)}(\alpha_{\delta-1+\lceil l/q \rceil})^{\oplus 2}.$$

5.6.3. Remark. Apparently in 5.6.2, $HF^+(-M)$ contains less information than the polynomial P, the above formula involves only the coefficients α_ℓ for $\ell \geq \delta - 1$. But this is not the case. Indeed, since the Alexander polynomial $\Delta(t)$ is symmetric, one gets that

$$t^{\mu-2}P(1/t) - P(t) = \frac{\delta(1+t^{\mu-1}) - (1+t+\cdots+t^{\mu-1})}{t-1},$$

hence $\alpha_{\mu-2-i} - \alpha_i$ is a universal number. In particular, from $\{\alpha_\ell\}_{\ell \geq \delta-1}$ one can recover P.

This shows that from $HF^+(-S^3_{-1/q}(K_f))$ one can recover both the integer q and the isotopy type of $K_f \subset S^3$. It looks that similar result is valid for general surgery coefficients as well.

5.6.4. Remark. In the above example it is striking a $\mathbb{Z}_2$-*symmetry* of the graded root and of $HF^+(-M)$. We will explain this for $a = 0$.

The point is that if the canonical characteristic element has only integral coefficients (i.e., if the graph is numerical Gorenstein), then all the theory associated with the integral cycles has a duality. This happens for example if $p = 1$; or if $q = 1$ and $2\delta - 2$ is divisible by p. In our case, in these situations, the function τ_0 is stable with respect to the $\mathbb{Z}_2$ action $i \mapsto 2t_0 + 2 - i$, i.e. $\tau_0(i) = \tau_0(2t_0 + 2 - i)$. This induces a symmetry of the root and of the Heegaard Floer homology.

But, for general p/q, the graphs are not 'numerical Gorenstein'. Even if they are, for general a, the symmetry may fail.

5.6.5. Example. Assume that $K_f \subset S^3$ is the torus knot $(4, 5)$ (i.e. $g = 1$ and $(p_1, q_1) = (4, 5)$). In this case $\delta = 6$, $\mu = 12$, S is generated by 4 and 5, hence

$$P(t) = 6 + 5t + 4t^2 + 3t^3 + 3t^4 + 3t^5 + 2t^6 + t^7 + t^8 + t^9 + t^{10}.$$

Then for some (p/q)-surgeries the corresponding graded roots are presented in the next Figure. [Here we did not draw all the vertices, only those ones which are either local minimums or supremums of pairs of local minimums; in fact, exactly these ones are given by the function τ_a. Also, the roots should be continued upward with one vertex in $\chi^{-1}(n)$ for each $n \geq 1$.]

Recall that when we compute the grading of HF^+, the value of τ_a is doubled, and then shifted by r_a. The corresponding values of r_a in these five cases are:

$$30,\ 71/4,\ 49/4,\ \frac{(p+10)^2 - p}{4p},\ 60.$$

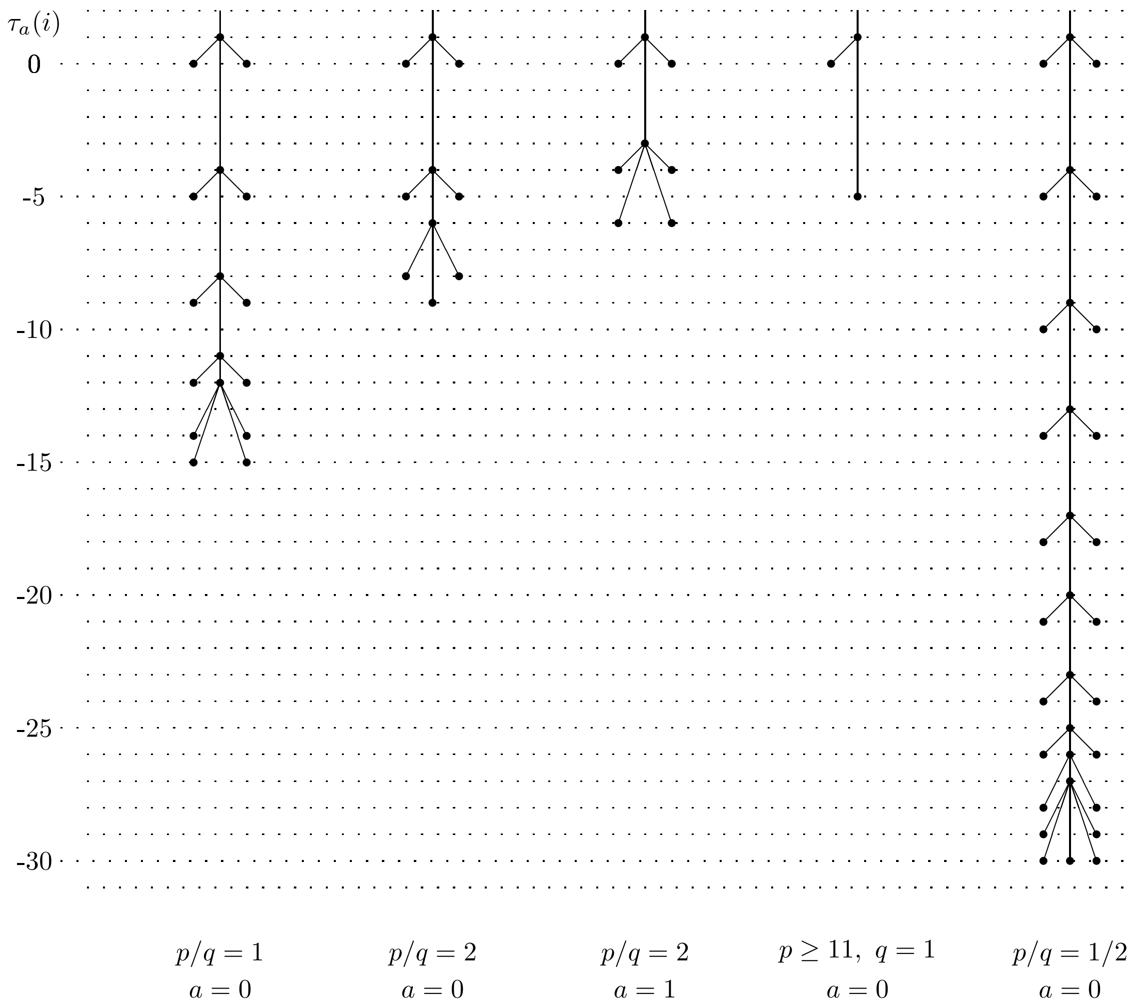
In particular, in the first and fifth case (i.e. when $p = 1$) one has $d(-M, \sigma_0) = 0$, as we expected: the Heegaard Floer homology are written in 5.6.1, resp. 5.6.2. In the second and third cases (i.e. when $p/q = 2$ and $a = 0, 1$) the $\mathbb{Z}[U]$-modules are:

$$\Big(\ \mathcal{T}^+_{-18} \oplus \mathcal{T}_{-16}(2)^{\oplus 2} \oplus \mathcal{T}_{-10}(1)^{\oplus 2} \oplus \mathcal{T}_0(1)^{\oplus 2}\ \Big)[71/4];$$

$$\Big(\ \mathcal{T}^+_{-12} \oplus \mathcal{T}_{-12}(3) \oplus \mathcal{T}_{-8}(1)^{\oplus 2} \oplus \mathcal{T}_0(1)^{\oplus 2}\ \Big)[49/4].$$

In the forth case it is

$$\mathcal{T}^+_{-10} \oplus \mathcal{T}_0(1)\Big[\frac{(p+1)^2 - p}{4p}\Big].$$



5.7 $S^3_{-p/q}(K)$ as Kulikov graph–manifold.

5.7.1. Notice that 5.2.4 and 4.4.4 imply that the geometric genus p_g of any normal surface singularity $(X, 0)$, whose link is diffeomorphic to $M = S^3_{-p/q}(K)$, satisfies the inequality *(SIn)* of 4.4.2, namely

$$p_g \leq -\mathbf{sw}(M, \sigma_{can}) - (K^2 + \#\mathcal{J})/8.$$

This, via 5.3.2 and 5.3.4 reads as follows:

5.7.2. Theorem. *Assume that the link M of $(X, 0)$ is $S^3_{-p/q}(K)$. Then*

$$p_g \leq \sum_{t\geq 0} \alpha_{[tp/q]}.$$

In the spirit of the second part *(SId)* of 4.4.2, (see also [37]), we expect that the above upper bound for p_g is an *optimal* topological upper bound (when we run all the possible analytic structures on $(X, 0)$). In order to be more cautious (for some warnings see [26]), we formulate it as a question:

5.7.3. Question. *Is it possible to find a special analytic singularity $(X, 0)$ with M as its link, such that*

$$p_g = \sum_{t\geq 0} \alpha_{[tp/q]} \ ?$$

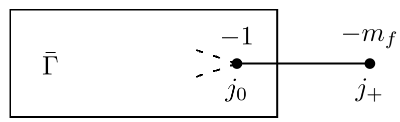
We invite the reader to formulate the similar inequality/identity for twisted line bundles, respectively for the other *spin*^{c} structures.

5.7.4. The above Question 5.7.3 can be made even more precise; i.e., we even propose a candidate for an analytic structure $(X, 0)$. The point is that there is a natural family of normal surface singularities, the so-called Kulikov singularities, with link M. In general, the fact, that on the topological type of a singularity one can put the analytic structure of a Kulikov singularity, can be determined topologically from the combinatorics of the plumbing graphs. The corresponding graphs are called 'Kulikov graphs' (Karras called them 'Kodaira graphs'). These particular graphs were studied intensively by Kulikov [16], Karras [15], Stevens [58] and others.

In the sequel, we exemplify the construction via M (in fact, proving that $\Gamma(M)$ is a Kulikov graph). The construction also provides a possible analytic germ $(X, 0)$ with special features.

Consider the plumbing graph:

$\Gamma_0(M)$:



Clearly, the corresponding plumbed 3-manifold is $S^3_0(K)$ (the manifold obtained by 0 surgery along K). The point is that this graph is negative semi-definite, hence by a theorem of Winters [67], there exists a holomorphic proper map $h : \mathcal{Y} \to D$ (where D is a small disc, and $\mathcal{Y}$ is a smooth surface) such that $h^{-1}(0)$ is a normal crossing divisor whose dual graph is $\Gamma_0(M)$, and the generic fiber of h is a smooth curve. Then one blows up k_1 generic points of the irreducible component of $h^{-1}(0)$ corresponding to the vertex j_+. In this way one creates k_1 (-1)–curves. Blow up $k_2 - 1$ generic points of one of these new curves, and keep the other $k_1 - 1$ unmodified; and continue this procedure. Then, the strict transform of $h^{-1}(0)$ together with all the exceptional divisors form a curve. Notice that the dual graph of all the irreducible components which are *not* (-1)–curves is exactly $\Gamma(M)$. These curves can be contracted providing a normal surface singularity $(X, 0)$ with dual resolution graph $\Gamma(M)$. The analytic structure of $(X, 0)$ has some nice properties (e.g. h induces a nice function on it, for details see e.g. [58]). A singularity constructed in this way is called Kulikov singularity.

5.7.5. Question. *Is it true that* $p_g = \sum_{t\geq 0} \alpha_{[tp/q]}$ *for a Kulikov singularity* $(X, 0)$ *with minimal good resolution graph* $G(M)$ *?*

6 Unicuspidal rational plane curves and $S^3_{-d}(K)$.

6.1 The semigroup distribution property.

6.1.1. The main objects of this section are irreducible rational projective plane curves in the complex projective space $\mathbb{P}^2$. It is a very difficult open problem to characterize the local embedded topological types of local singular germs which can be realized as the singularities of such a projective curve C of degree d. This problem has a long and rich history providing many interesting compatibility properties connecting local invariants of the singular germs $\{(C, p_i)\}_i$ with some global invariants of C — like its degree, or the log-Kodaira dimension of $\mathbb{P}^2 \setminus C$, etc. For a (non-complete) list of some of these restrictions, see e.g. [8, 9, 10] and the references therein. The simplest one is the genus formula: the sum of the Milnor numbers of the local plane curve singularities $\{(C, p_i)\}_i$ should be $(d-1)(d-2)$. (But, this is far to be enough for the characterization.)

The article [8] proposes a new compatibility property — valid for rational cuspidal curves C. (A 'cusp' means a locally irreducible singularity, C is 'cuspidal' if it has only cusps.) Its formulation is the following. Consider a collection $(C, p_i)_{i=1}^{\nu}$ of cusps, let $\Delta_i(t)$ be their characteristic polynomials, and set $\Delta(t) := \prod_i \Delta_i(t)$. Its degree is 2δ, where δ is the sum of the delta-invariants of the singular points. (By the above mentioned genus formula: $2\delta = (d-1)(d-2)$.) Write $\Delta(t)$ as $1 + (t-1)\delta + (t-1)^2 Q(t)$ for some polynomial $Q(t)$, cf. 5.2.2. Let c_l be the coefficient of $t^{(d-3-l)d}$ in $Q(t)$ for any $l = 0, \ldots, d-3$. (In the notations of 5.2.2 one has $c_l = \alpha_{d(d-3-l)}$.)

6.1.2. Conjecture A. [8] *Let $(C,p_i)_{i=1}^{\nu}$ be a collection of cusps, such that $2\delta=(d-1)(d-2)$ for some integer d. If $(C,p_i)_{i=1}^{\nu}$ can be realized as the local singularities of a degree d (automatically rational and cuspidal) projective plane curve, then*

$$c_l \le (l+1)(l+2)/2 \;\; \text{for all} \;\; l=0,\dots,d-3.$$

In fact, the integers $n_l := c_l-(l+1)(l+2)/2$ are symmetric: $n_l=n_{d-3-l}$; and $n_0=n_{d-3}=0$ automatically. We also mention that strict inequality (with $\nu>1$) may occur, cf. [8].

Let $\bar\kappa(\mathbb{P}^2\setminus C)$ denote the logarithmic Kodaira dimension of $\mathbb{P}^2\setminus C$. The main result of [8] is:

6.1.3. Theorem. [8] *If $\bar\kappa(\mathbb{P}^2\setminus C)\le 1$, then the above conjecture A is true (with $n_l=0$).*

On the other hand, rather surprisingly, in the *unicuspidal* case one can show the following.

6.1.4. Proposition. [8] *If $\nu=1$ then $c_l\ge (l+1)(l+2)/2$ for $0\le l\le d-3$.*

Therefore, conjecture 6.1.2 in this case can be reformulated as follows:

6.1.5. Conjecture B1. *With the notations of 6.1.2, if $\nu=1$, then*

$$c_l = (l+1)(l+2)/2 \;\; \text{for all} \;\; l=0,\dots,d-3.$$

If $\nu=1$, the characteristic polynomial Δ of $(C,p)\subset(\mathbb{P}^2,p)$ is a complete embedded topological invariant of this germ, similarly as the semigroup $S\subset\mathbb{N}$. Hence, one can reformulate conjecture B1 in terms of S and d. It turns out that the collection of vanishings of all the coefficients n_l is replaced by a very precise and mysterious distribution of the elements of the semigroup with respect to the (half–open) intervals $I_l := (\,(l-1)d, ld\,]$:

6.1.6. Conjecture B2. *Assume that $\nu=1$. Then for any $l>0$, the interval I_l contains exactly $\min\{l+1,d\}$ elements from the semigroup.*

In other words, for every rational unicuspidal plane curve C of degree d, the above conjecture is equivalent with the conjectural identity $D(t)\equiv 0$, where:

$$D(t) := \sum_{k\in S} t^{\lceil k/d\rceil} - \Big(1+2t+\cdots+(d-1)t^{d-2}+d(t^{d-1}+t^d+t^{d+1}+\cdots)\Big). \qquad (DP)$$

For an explanation of the equivalences of conjectures B1 and B2, see 6.2.4. Here we only recall the key connection between the coefficients c_l and the semigroup S:

$$c_l = \#\{k\in S\,;\, k\le ld\}.$$

This follows from 5.2.2 and from the fact that for $0\le k<\mu$ one has: $k\in S$ if and only if $\mu-k-1\notin S$.

6.2 The semigroup distribution property and surface singularities.

6.2.1. Superisolated singularities. The theory of isolated hypersurface surface singularities 'contains' in a canonical way the theory of complex projective plane curves via the family of *superisolated* singularities. These singularities were introduced by I. Luengo in [25].

A hypersurface singularity $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$, $f=f_d+l^{d+1}$ (where f_d is homogeneous of degree d and l is linear) is superisolated if the projective plane curve $C:=\{f_d=0\}\subset\mathbb{P}^2$ is reduced, and its singularities $\{p_i\}_{i=1}^{\nu}$ are not situated on $\{l=0\}$. The equisingular type of f depends only on f_d, i.e. only on the projective curve $C\subset\mathbb{P}^2$. In particular, all the invariants (of the equisingular type) of f can be determined from the invariants of the pair $(\mathbb{P}^2,C)$.

6.2.2. *Assume in the sequel that C is irreducible, rational with exactly one singular point (C,p).*

In the next discussion we follow [26, 8]. Let $\mu=2\delta$, S and Δ be the local invariants of (C,p). Set $\bar\Delta(t):=t^{-\delta}\Delta(t)$. Let $K\subset S^3$ be the local embedded link of (C,p). Then, one shows that $M=S^3_{-d}(K)$; in particular, $H_1(M,\mathbb{Z})=\mathbb{Z}_d$.

In fact, if we blow up the maximal ideal of the singular point of $\{f = 0\}$ we get its minimal resolution, which contains one irreducible exceptional divisor, which is isomorphic to C and has self–intersection $-d$. From this picture the above identification of M also follows. But from this fact one also has that the invariant $K^2 + \#\mathcal{J}$ (of M) equals $1 - d(d-2)^2$, hence it depends only on d. One also computes $p_g = d(d-1)(d-2)/6$.

Before we continue our discussion, we recall that the Property *(SId)* (conjectured for 'nice' singularities), applied for the germ f and the canonical *spin*^{c} structure reads as follows:

6.2.3. Property SWC. [37] $\quad -\mathbf{sw}(M, \sigma_{can}) - (K^2 + \#\mathcal{J})/8 = p_g.$

In our case, since p_g and $K^2 + \#\mathcal{J}$ depend only on d, the validity of SWC (6.2.3) would impose serious restriction on the link $M = S^3_{-d}(K)$, or equivalently on the local knot $K \subset S^3$, which can be measured by the local invariants Δ or S. In this subsection we will explain this fact via the interpretation of $\mathbf{sw}(M, \sigma_{can})$ by the Reidemeister-Turaev torsion, cf. 2.3.4. One shows, cf. [26], that

$$\mathcal{T}_{M,\sigma_{can}}(1) = \frac{1}{d} \sum_{\xi^d = 1 \neq \xi} \frac{\Delta(\xi)}{(\xi - 1)^2} \quad \text{and} \quad \lambda(M) = -\frac{\bar{\Delta}(t)''(1)}{2} + \frac{(d-1)(d-2)}{24}.$$

Consider also (motivated partly by the above formula of the torsion):

$$R(t) := \frac{1}{d} \sum_{\xi^d = 1} \frac{\Delta(\xi t)}{(1 - \xi t)^2} - \frac{1 - t^{d^2}}{(1 - t^d)^3}.$$

Similarly,

$$N(t) := \sum_{l=0}^{d-3} \Big(c_l - \frac{(l+1)(l+2)}{2} \Big) t^{d-3-l}; \quad \text{and} \quad D(t) := \sum_{k \in S} t^{\lfloor k/d \rfloor} - \frac{1 - t^d}{(1-t)^2}.$$

Notice that this $D(t)$ agrees with the one defined in (DP) in the previous subsection. In [8] the following facts are verified:

- $R(t) = D(t^d)/(1 - t^d) = N(t^d)$;
- $N(t)$ (hence $R(t)$ too) has non-negative coefficients;
- $R(1) = -p_g - \mathbf{sw}(M, \sigma_{can}) - (K^2 + \#\mathcal{J})/8$.

In particular, we have the equivalence of the 'Seiberg-Witten invariant conjecture' with the 'semigroup distribution property':

6.2.4. Theorem. *Under the assumptions (6.2.2) the following facts are equivalent:*
(a) $R(1) = 0$*, i.e. (the conjectured) Property SWC (6.2.3) is true (for the above germ f);*
(b) $R(t) \equiv 0$*;*
(c) $N(t) \equiv 0$*, i.e. Conjecture B1 (6.1.5) is true;*
(d) $D(t) \equiv 0$*, i.e. Conjecture B2 (6.1.6) is true.*

6.3 The semigroup distribution property and graded roots/Heegaard Floer homology.

6.3.1. Assume that we are in the situation of 6.2.2. In this subsection we will compare the invariants of the link M with the corresponding invariants of the Seifert 3–manifold $\Sigma(d, d, d+1)$ — this is the link of the hypersurface Brieskorn singularity $x^d + y^d + z^{d+1} = 0$. This connection is very surprising, the possible conceptual relationship between these two spaces (together, hopefully, with the proof of conjectures listed in 6.2.4) will be explained in a forthcoming manuscript.

6.3.2. Theorem.[10] *The following facts are equivalent:*
(a) Conjecture B1 (6.1.5) is true,
(b) The canonical graded roots of $S^3_{-d}(K)$ *and* $\Sigma(d, d, d+1)$ *are the same.*

(c) The canonical Heegaard-Floer homologies of $-S^3_{-d}(K)$ and $-\Sigma(d,d,d+1)$ are the same modulo a grading shift, namely:

$$HF^+(-S^3_{-d}(K),\sigma_{can})[1-d(d-2)^2] = HF^+(-\Sigma(d,d,d+1),\sigma_{can})[-d(d-1)(d-3)].$$

(d)

$$\Big(\mathbf{sw}(M,\sigma_{can}) + \frac{K^2+\#\mathcal{J}}{8}\Big)\Big|_{M=S^3_{-d}(K)} = \Big(\mathbf{sw}(M,\sigma_{can}) + \frac{K^2+\#\mathcal{J}}{8}\Big)\Big|_{M=\Sigma(d,d,d+1)}.$$

The proof is given in several steps. The main point is that both 3–manifolds $S^3_{-d}(K)$ and $\Sigma(d,d,d+1)$ are *almost rational*, cf. (2.7). In particular, in both cases, the canonical graded root can be determined via the 'τ-construction' 3.5.6. In the first case this is done explicitly in 5.3.2, while for the second case, see [34], last section.

6.3.3. Example. Let us rewrite 5.3.2 for the case $M = S^3_{-d}(K)$ and for the canonical $spin^c$ structure: $a=0$, $q=q'=1$, $p=d$. Set $c_l := \alpha_{(d-3-l)d}$ and define $\tau : \{0,1,\ldots,2d-4\} \to \mathbb{Z}$ by

$$\tau(2l) = \tfrac{l(l-1)}{2}d - l(\delta-1), \qquad \tau(2l+1) = \tau(2l+2) + c_{d-3-l}.$$

Then $(R_{can},\chi_{can}) = (R_\tau,\chi_\tau)$.

6.3.4. Example. Consider the Seifert manifold $\Sigma(d,d,d+1)$. Its canonical graded root is the following (cf. [34]). For any $0\leq l\leq d-3$ write $c^u_l := (l+1)(l+2)/2$, and $2\delta := (d-1)(d-2)$. Then define $\tau^u : \{0,1,\ldots,2d-4\} \to \mathbb{Z}$ by

$$\tau^u(2l) = \tfrac{l(l-1)}{2}d - l(\delta-1), \qquad \tau^u(2l+1) = \tau^u(2l+2) + c^u_{d-3-l}.$$

Then $(R_{can},\chi_{can}) = (R_{\tau^u},\chi_{\tau^u})$.

Notice the shocking similarities of 6.3.3 and 6.3.4: the graded roots associated with $S^3_{-d}(K)$ and $\Sigma(d,d,d+1)$ coincide exactly when $c_l = c^u_l$ for all l, a system of equalities subject of Conjecture B1. This proves $(a) \Leftrightarrow (b)$. $(b) \Rightarrow (c)$ follows from 3.5.2, since in both cases

$$HF^+(-M,\sigma_{can})[(K^2+\#\mathcal{J})/4] = \mathbb{H}(R_{can},\chi_{can}),$$

where the shift in the case of $M = S^3_{-d}(K)$ is $K^2+\#\mathcal{J} = 1-d(d-2)^2$, while for $M=\Sigma(d,d,d+1)$ is $K^2+\#\mathcal{J} = -d(d-1)(d-3)$. This, and the corresponding definitions show $(c) \Rightarrow (d)$.

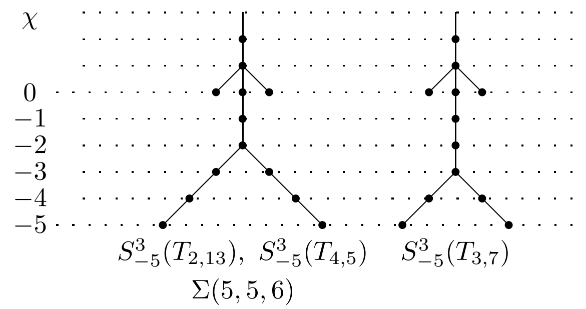
Since the Property SWC (6.2.3) is true for the Brieskorn singularity $f_{BR} := x^d+y^d+z^{d+1}$ (cf. [38]), and the geometric genus of the superisolated singularity f equals the geometric genus of f_{BR} (both equal $d(d-1)(d-2)/6$), this last identity (d) is also equivalent with the validity of the Property SWC (6.2.3) for f — a fact, which is equivalent with (a) by 6.2.4.

6.3.5. Remark. For $(d) \Rightarrow (b)$ one can also argue as follows. Regarding the Seiberg-Witten invariant of $M = S^3_{-d}(K)$, by 5.3.2 and 5.3.4 one has

$$-\mathbf{sw}(M,\sigma_{can}) - \frac{K^2+\#\mathcal{J}}{8} = \sum_{l\geq 0}\tau(2l+1)-\tau(2l+2) = \sum_{l\geq 0} c_l,$$

and there is a similar formula for $M=\Sigma(d,d,d+1)$ with the obvious replacements.

6.3.6. Example. Assume that $d=5$ and C is unicuspidal whose singular point has only one Puiseux pair (a,b) with $a<b$. Then by the genus formula the possible values of (a,b) are $(4,5)$, $(3,7)$ and $(2,13)$. It turns out that the first and the third cases can be realized, while the second not. The corresponding canonical graded roots (together with the root of $\Sigma(5,5,6)$) are drawn in the next picture.



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